10.2 Series:

Dangerous bend. We now turn our attention from the sequence $\{a_n\}_{n\geq 0}$ to the sum

$$a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + \dots$$

The sum is referred to as a series. Since there are infinitely many terms it is possible that the sum does not exist (divergent) but in many interesting cases it is convergent.

Motivation. Consider the process where a white square is divided into 25 equal squares and one of these squares is blackened. Then, each of the remaining white squares is divided into 25 equal squares and one of them is blackened. If the process is continued indefinitely then how many steps are required before 80% of the original white square is blackened.

After one step 1/25(=0.04) is blackened. After 2 steps

$$\frac{1}{25} + \frac{24}{25^2} \qquad (= 0.0784)$$

is blackened and after 3 steps

$$\frac{1}{25} + \frac{24}{25^2} + \frac{24^2}{25^3} \qquad (= 0.115264)$$

is blackened because of the 25^2 small squares $25 + 24$ are already black and so $25^2 - 25 - 24 = 24^2$ will participate in the third step. Thus a sum represents how much of the square is black at each step. To find out the outcome of the experiment after many iterations we want to look at the asymptotic behavior of the sum that we look at the series

$$\sum_{n=0}^{\infty} \frac{24^n}{25^{n+1}}$$

Roughly 38 iterations should make the square 80% black.

Zeno's Paradox: A hare and a tortoise are racing. The tortoise has a 50 yard head start and we assume that the the hare runs 100 yards/minute and the tortoise runs 50 yards per minute. When the hare reaches where the tortoise starts (50 yd) the tortoise has move on 25 yards. When the hare moves on 25 more yards then the tortoise has moved on 12.5 more yards. When will the hare pass the tortoise?

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

Zeno knew that they both reach the 100 yd mark in 1 minute. On the other hand he thought that if you added up infinitely many positive numbers you must get infinity. But is that so? $1/2 + 1/4 + 1/8 = 1 - 1/8$. The sum above is 1 minus the last term which means very close to 1 minute just as we know the correct answer to be.

Definition. A series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$ has nth partial sum

$$s_n = a_1 + a_2 + a_3 + \dots + a_n$$

If the sequence s_n of partial sums converges then the series is said to be convergent with limit (sum) $\sum_{n=1}^{\infty} a_n$ and otherwise the series is said to be divergent.

Example: Evaluate the series

$$\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10^4}+\frac{3}{10^5}+\ldots = 3\sum_{n=1}^{\infty} 10^{-n}$$

Well this is just the decimal number .3333333333333333333333... etc and so we know the answer is 1/3.

Geometric Series:

$$\sum_n = 1^{\infty} ar^{n-1}$$

The partial sum s_5 in this case is

$$s_5 = a + ar + ar^2 + ar^3 + ar^4$$

Observe

$$\begin{aligned} s_5 &= a + \quad ar + ar^2 + ar^3 + ar^4 \\ rs_5 &= \qquad\quad ar + ar^2 + ar^3 + ar^4 + ar^5. \end{aligned}$$

Subtract: $(1-r)s_5 = a(1-r^5)$. Provided that $r \neq 1$

$$a + ar + ar^2 + ar^3 + ar^4 = \frac{a(1-r^5)}{1-r}$$

Similar reasoning shows that

$$a + ar + ar^2 + ar^3 + \ldots + ar^n = \frac{a(1-r^{n+1})}{1-r}$$

What happens as $n \to \infty$? If $|r| < 1$ then convergence to

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad \text{if } |r| < 1$$

and the series diverges if $|r| \geq 1$.

Example. $1 + 1/2 + 1/4 + 1/8 + 1/16 + \ldots + 1/2^n + \ldots = 1/(1-0.5) = 2$

Example. $1/2 + 1/4 + 1/8 + 1/16 + \ldots + 1/2^n + \ldots = 1$ which agrees with the Zeno Paradox example.

Example. $\sum_{n=1}^{\infty} 3/10^{n-1} = 3/(1-1/10) = 10/3$. Therefore $3/10 + 3/10^2 + 3/10^3 + 3/10^4 + \ldots = 10/3 - 3 = 1/3$

Example: For what values of x does the series below converge? Evaluate the series.

$$\sum_{n=0}^{\infty} \frac{5(x+1)^n}{2^n}$$

Solution. This is the geometric series with $r = (x+1)/2$. Need $|r| < 1$ or $|x+1| < 2$ so that $-3 < x < 1$. the series converges to

$$\sum_{n=0}^{\infty} \frac{5(x+1)^n}{2^n} = \frac{5}{1-(x+1)/2} = \frac{10}{2-(x+1)} = \frac{10}{1-x}.$$

Harmonic Series: $\sum_{n=1}^{\infty} \frac{1}{n}$ This series does not converge because the terms go to 0 too slowly.

$$\begin{aligned} & 1 + 1/2 + [1/3 + 1/4] + [1/5 + 1/6 + 1/7 + 1/8] \\ + \quad & [1/9 + 1/10 + 1/11 + 1/12 + 1/13 + 1/14 + 1/15 + 1/16] + \dots \end{aligned}$$

and every bracketed expression is at least 1/2. The terms from the 2^n+1th to 2^{n+1}th add to a sum at least 1/2 so that the entire sum must be ∞. The harmonic series diverges.

Telescoping Series: Sum the series

$$\frac{1}{3} - \frac{1}{5} + (\frac{1}{5} - \frac{1}{7}) + (\frac{1}{7} - \frac{1}{9}) + + \dots + (\frac{1}{2n+1} - \frac{1}{2n+3}) + \dots = \frac{1}{3}$$

Example. Evaluate

$$\sum_{n=1}^{\infty} \frac{2}{(2n+1)(2n+3)}$$

Solution. Apply partial fractions.

$$\frac{2}{(2n+1)(2n+3)} = \frac{A}{(2n+1)} + \frac{B}{(2n+3)}$$

Find $A = 1$, $B = -1$ so that

$$\frac{2}{(2n+1)(2n+3)} = \frac{1}{(2n+1)} - \frac{1}{(2n+3)}$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2}{(2n+1)(2n+3)} &= 2/15 + 2/35 + \dots \\ &= [1/3 - 1/5] + [1/5 - 1/7] + [1/7 - 1/9] + [1/9 - 1/11] + \dots \\ &= 1/3 \end{aligned}$$

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Test for Divergence If $\lim_{n\to\infty} a_n \neq 0$ or the limit does not exist then the series

$$\sum_{n}^{\infty} a_n \text{ diverges.}$$