**10.1 Sequences:**

A sequence is a function $a$ defined on the integers, $n = 1, 2, 3, \ldots$. The function values are written $a(n) = a_n$.

Example: $a_1 = 1$, $a_2 = 1/2$, $a_3 = 1/3$, $a_4 = 1/4$ and so on. It is briefer to give a formula $a_n = 1/n$.

Example: $a_n = 2^{-n}$, $n \geq 0$. This is an abbreviation of $a_0 = 2^{-0} = 1$, $a_1 = 1/2$, $a_2 = 1/4$, $a_3 = 1/8$ and so on: $1, 1/2, 1/4, 1/8, 1/16, 1/32, \ldots$. It is often useful to graph. The graph just consists of dots.

Example (Fibonacci sequence): 1,1,2,3,5,8,13,21,34,55,89,144,233,....

Convergence: A sequence $\{a_n\}_{n\geq 1}$ converges to a *limit* $L$ if for every $\epsilon > 0$ there is $N > 0$ so that $n \geq N$ implies

$$|a_n - L| < \epsilon$$

In symbols:

$$\lim_{n\to\infty} a_n = L$$

We have already seen $\lim_{x\to\infty} f(x) = L$ and this is the same. Picture

Example:

1. $\lim_{n\to\infty} 1/n = 0$
2. $\lim n{\to}\infty \frac{2n^2-n}{3n^2+11} = 2/3$
3. $\lim n{\to}\infty \frac{2n^5-n}{3n^2+11} = \infty$

Rules: Recall that you can add, subtract and multiply limits and even divide provided the limit on bottom is not 0. Also if $f(x)$ is continous at $L = \lim_{n\to\infty} a_n$ then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

Example: $\lim\limits_{n\to\infty} \sqrt{\frac{4n-3}{9n+2}} = 2/3$ because the square root function is continuous.

Common Limits

1. $\lim_{n\to\infty} \frac{\ln n}{n} =$ __________
   Check: L'Hopital's rule:
$$\lim_{n\to\infty} \frac{\ln n}{n} = \left(\frac{\infty}{\infty}\right) = \lim n \to \infty \frac{1/n}{1} = 0.$$
2. $\lim_{n\to\infty} n^{1/n} = 1$ for just exponentiate the previous limit.
3. $\lim_{n\to\infty} x^{1/n} = 1$ for $x > 0$
4. $$\lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } |x| < 1 \\ 1 & \text{if } x = 1 \\ \infty & \text{if } x > 1 \\ \text{DNE} & \text{if } x \leq -1 \end{cases}$$
5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n =$ __________
6. $\lim n \to \infty \frac{x^n}{n!} = 0$

The fifth limit is best evaluated by taking ln of the left side and taking the limit
$$\lim_{n\to\infty} \ln \left(1 + \frac{x}{n}\right)^n = \lim_{n\to\infty} \frac{\ln\left(1 + \frac{x}{n}\right)}{1/n} (= \frac{0}{0}) = \lim_{n\to\infty} \frac{\frac{1}{1+\frac{x}{n}} \frac{-x}{n^2}}{-1/n^2} = x$$
by L'Hopital's rule. so that
$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$$
The sixth limit, follows by noting $x$ is fixed: $|x| < M$ for some integer $M$ Choose $n > 2M$.
$$\frac{x^n}{n!} = \frac{x^{2M}}{(2M)!} \frac{x^{n-2M}}{n(n-1)(n-2)\dots(2M+1)}$$
The last factor goes to 0 as $n \to \infty$ because there are $n - 2M$ factors all smaller than $1/2$.

**Monotonic Sequences** A sequence $a_n$ is increasing if $a_n \leq a_{n+1}$ for every $n$. Alternatively $a_n$ is decreasing if $a_n \geq a_{n+1}$ for every $n$. A sequence which is either increasing or decreasing (can't be both!) is *monotonic.*

A sequence $(a_n)_{n\geq 1}$ is *bounded* if there is a constant $M$ so that $|a_n| \leq M$.

Theorem 10. A bounded monotonic sequence is convergent.

Proof. Suppose that the sequence is decreasing (book does increasing case). Since the sequence is bounded it is "bounded below." Therefore there is a number $L$ so that $a_n \geq L$ for all $N$. Suppose that $L$ is the greatest lower bound. Then,if $\epsilon > 0$ is any given positive number then $L + \epsilon$ is not a lower bound. This says that there is $N$ so that $a_N < L + \epsilon$ but this says that $a_n < L + \epsilon$ for all $n$.

Example: $a_1 = 1/2$ $a_2 = 3/4$, $a_3 = 7/8$ and so on. Then $a_n = 1 - 1/2^n$ is an increasing sequence bounded above by 1 and convergent to ?