2. Measurability

It is an objective of this Chapter to define the integral of a function. If f is a nonnegative function defined on an interval in the real line then the integral should be able to tell us the area under the graph of f just as the Riemann integral does in calculus. It will tell us much more but even in this limited context we shall discover that f must be restricted: the notion of area under the graph of f will not make sense for every f and we will be forced to restrict the class of functions considered. We introduce in this Section the concept of a "measurable" function. We shall see that f must be measurable if we are to make sense of the notion of area under the graph. Of course this difficulty arises already in the case of Riemann integration: the Riemann integral is not defined for arbitrary functions f. We shall clarify this remark and discuss the Riemann integral and its relation to the "Lebesgue integral," introduced here, at the end of this Chapter.

Definition: Let Ω be a set and $\mathcal{F}$ be a σ-algebra of subsets of Ω. The we shall refer to $(\Omega, \mathcal{F})$ as a *measurable space.* The sets in $\mathcal{F}$ will be referred to as *measurable sets.*

Definition: Suppose $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ are two measurable spaces. Then a mapping $f : \Omega_1 \to \Omega_2$ is *measurable*, with respect to $\mathcal{F}_1$ and $\mathcal{F}_2$ if $f^{-1}(B) \in \mathcal{F}_1$ whenever $B \in \mathcal{F}_2$. We shall sometimes say f is measurable function from $(\Omega_1, \mathcal{F}_1)$ to $(\Omega_2, \mathcal{F}_2)$ to clarify the choice of σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$. Unless otherwise specified a function $f : \Omega_1 \to \mathbb{R}^n$ is said to be *measurable* or *Borel measurable* if f is measurable from $(\Omega_1, \mathcal{F}_1)$ to $(\mathbb{R}^n, \mathcal{B})$.

An elementary example of a measurable function is given by the characteristic function χ_A of a set A . Then χ_A is measurable as a mapping from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B})$ if and only if $A \in \mathcal{F}$, that is if and only if A is a measurable set. Further examples of measurable functions will be apparent once some elementary properties have been established.

LEMMA 2.1. *Suppose that $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$ and $(\Omega_3, \mathcal{F}_3)$ are three measurable spaces and that $f : \Omega_1 \to \Omega_2$ and $g : \Omega_2 \to \Omega_3$ are two mappings. If f and g are both measurable then the composed function $g \circ f$ is measurable from $(\Omega_1, \mathcal{F}_1)$ to $(\Omega_3, \mathcal{F}_3)$.*

Proof: We must show that, for an arbitrary set $C \in \mathcal{F}_3$, $(g \circ f)^{-1}(C) \in \mathcal{F}_1$. It suffices to show that

$$(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C)) \qquad (2.2.1)$$

because $g^{-1}(C) \in \mathcal{F}_2$ because g is measurable and so $f^{-1}(g^{-1}(C)) \in \mathcal{F}_1$ because f is measurable. The verification of (2.2.1) is a straightforward exercise in checking the equality of sets. □

PROPOSITION 2.2. *Suppose that $f : \Omega_1 \to \Omega_2$ is a function and $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-algebras on Ω_1 and Ω_2 respectively. Suppose further that $\mathcal{S}$ is a system of generators of $\mathcal{F}_2$, which is to say $\mathcal{F}_2 = \sigma(\mathcal{S})$. Then f is measurable from $(\Omega_1, \mathcal{F}_1)$ to $(\Omega_2, \mathcal{F}_2)$ if and only if $f^{-1}(B) \in \mathcal{F}_1$ for every $B \in \mathcal{S}$.*

Proof: It is obvious that, if f is measurable then $f^{-1}(B) \in \mathcal{F}_1$ for every $B \in \mathcal{S}$ simply because $\mathcal{S} \subseteq \mathcal{F}_2$. Conversely, suppose $f^{-1}(B) \in \mathcal{F}_1$ for every $B \in \mathcal{S}$. Let $\mathcal{T} = \{B \in \mathcal{F}_2 : f^{-1}(B) \in \mathcal{F}_1\}$ so that $\mathcal{T} \supseteq \mathcal{S}$. We shall show that $\mathcal{T}$ is a σ-algebra which implies that $\mathcal{T} = \mathcal{F}_2$ and so, by the definition of $\mathcal{T}$, f is measurable. This will complete the proof.

We check therefore that $\mathcal{T}$ is a σ-algebra. Certainly $\emptyset \in \mathcal{T}$, because $f^{-1}(\emptyset) = \emptyset \in \mathcal{F}_1$. We check next that if $B \in \mathcal{T}$ then $f^{-1}(B) \in \mathcal{F}_1$ and so $f^{-1}(B^c) = f^{-1}(B)^c \in \mathcal{F}_1$ because $\mathcal{F}_1$ is closed under taking complements. (Recall $f^{-1}(B^c) = f^{-1}(B)^c$ by an Exercise.) We check finally that if $(B_n)_{n\in\mathbb{N}}$ is a sequence of sets in $\mathcal{T}$ so that $f^{-1}(B_n) \in \mathcal{T}$ for each $n \in \mathbb{N}$ then

$$f^{-1}(\cup_{n\in\mathbb{N}} B_n) = \cup_{n\in\mathbb{N}} f^{-1}(B_n) \in \mathcal{F}_1$$

because $\mathcal{F}_1$ is a σ-algebra. This shows that $\mathcal{T}$ is closed under countable unions and is therefore a σ-algebra and the proof is complete. □

Remark: This set $\mathcal{T}$ in the proof constitute "good" sets and the argument that there are many good sets is an instance of what Ash calls the *good sets principle* in his text page 5.

Remark: The above proof uses a special case of the following observation: If f is a mapping $f : \Omega_1 \to \Omega_2$ and if $(B_\alpha)_{\alpha\in\mathcal{I}}$ is a collection of subsets of Ω_2 indexed by a set $\mathcal{I}$ then

$$\begin{aligned} f^{-1}(\cup_{\alpha\in\mathcal{I}} B_\alpha) &= \cup_{\alpha\in\mathcal{I}} f^{-1}(B_\alpha) \\ f^{-1}(\cap_{\alpha\in\mathcal{I}} B_\alpha) &= \cap_{\alpha\in\mathcal{I}} f^{-1}(B_\alpha). \end{aligned}$$

On the other hand if $(A_\alpha)_{\alpha\in\mathcal{I}}$ is a collection of subsets of Ω_1 then

$$f(\cup_{\alpha\in\mathcal{I}} A_\alpha) = \cup_{\alpha\in\mathcal{I}} f(A_\alpha)$$

but it may happen that $f(\cap_{\alpha\in\mathcal{I}} A_\alpha) \neq \cap_{\alpha\in\mathcal{I}} f(A_\alpha)$.

COROLLARY 2.3. *A continuous function $f : \mathbb{R}^m \to \mathbb{R}^n$ is Borel measurable.*

Of course the understood σ-algebras here are the Borel σ-algebra $\mathcal{B}(\mathbb{R}^m)$ and $\mathcal{B}(\mathbb{R}^n)$.

Proof: Since the $\mathcal{B}(\mathbb{R}^n)$ is generated by the open sets, it suffices to show that $f^{-1}(U) \in \mathcal{B}(\mathbb{R}^m)$ for an arbitrary open set U. The result will then follow from the Proposition. However f is continuous means that $f^{-1}(U)$ is open whenever U is and so $f^{-1}(U) \in \mathcal{B}(\mathbb{R}^m)$. □

The same reasoning applies in a more general setting.

COROLLARY 2.4. *If f is a continuous function from one topological space (X, Σ) to another $(Y, \mathcal{T})$ then f is measurable from $(X, \mathcal{B}(X))$ to $(Y, \mathcal{B}(Y))$ where $\mathcal{B}(X)$ is the Borel σ-algebra which is generated by the open sets Σ of X and similarly $\mathcal{B}(Y)$ is generated by the open sets $\mathcal{T}$.*

PROPOSITION 2.5. *Suppose that $(\Omega, \mathcal{F})$ is a measurable space and f_1, f_2, ... f_m are m Borel measurable real valued functions, $f_j : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$, for $1 \leq j \leq m$. Suppose that $g : (\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable. Then*

$$\phi(x) = g(f_1(x), f_2(x), \ldots, f_m(x))$$

defines a measurable function $\phi : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

We shall set aside the proof of the Proposition until later and consider its consequences.

COROLLARY 2.6. *If $f_j : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$, $j = 1, 2$ are Borel measurable functions, then $f_1 + f_2$ is also Borel measurable.*

PROOF. This is an application of the Proposition with $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x_1, x_2) = x_1 + x_2$. Because g is continuous it is Borel measurable. □

COROLLARY 2.7. *If $f_j : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$, $j = 1, 2$ are Borel measurable functions, then the product $f_1 f_2$ is also Borel measurable. In particular, if k is a real constant then $k f_2$ is Borel measurable.*

PROOF. In this case $g(x, y) = xy$. The special case follows by defining $f_1(x) = k$ which makes f_1 measurable because it is continuous. □

COROLLARY 2.8. *If $f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$, is a Borel measurable function then $|f|$ is also Borel measurable.*

PROOF. In this case $g(x) = |x|$. □

COROLLARY 2.9. *If $f_j : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$, $j = 1, 2$ are Borel measurable functions, and if $f_2(x) \neq 0$ for all $x \in \Omega$ then the ratio f_1/f_2 is also Borel measurable.*

PROOF. In this case we define

$$g(x,y) = \begin{cases} x/y & \text{if} \quad y \neq 0 \\ 0 & \text{if} \quad y = 0 \end{cases}$$

so that g is defined on all of $\mathbb{R}^2$ but of course it is not continuous. We shall check however that g is measurable, as a mapping from $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ to $(\mathbb{R}, \mathcal{B})$. It suffices to show that $g^{-1}(-\infty, a) \in \mathcal{B}(\mathbb{R}^2)$ for any real a by Proposition 2.2. If $a < 0$ then $g^{-1}(-\infty, a) = \{(x,y) \in \mathbb{R}^2 : x/y < a\}$ is easily seen to be open in $\mathbb{R}^2$ and hence in $\mathcal{B}(\mathbb{R}^2)$. If $a \geq 0$ then $g^{-1}(-\infty, a) = \{(x,y) \in \mathbb{R}^2 : x/y < a\} \cup \{(x,y) : y = 0\}$ which is the union of an open and a closed set and is therefore in $\mathcal{B}(\mathbb{R}^2)$. This proves that g is measurable and so the above Proposition applies which completes the proof. □

Exercise: Show that if $f_j : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$, $j = 1, 2$ are Borel measurable functions then $\max\{f_1, f_2\}$ and $\min\{f_1, f_2\}$ are also measurable. Remark: $\max\{a, b\} = (|a - b| + a + b)/2$ for any reals a and b.

It remains to establish the Proposition.

Proof of the Proposition: It suffices to show that the mapping, f say, defined by $f(x) = (f_1(x), f_2(x), \ldots f_m(x))$ is measurable as a mapping from $(\Omega, \mathcal{F})$ to $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$ because the composition of measurable functions is measurable. To verify f is measurable, it suffices to show that, for an arbitrary $a = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$, $f^{-1}(\{x_1 < a_1, x_2 < a_2, \ldots, x_m < a_m\}) \in \mathcal{F}$ by Proposition 2.2. However

$$f^{-1}(\{x_1 < a_1, x_2 < a_2, \ldots, x_m < a_m\}) = \cap_{1 \leq j \leq m} f_j^{-1}(-\infty, a_j)$$

and as the intersection of m measurable sets, this set is, itself measurable. □

DEFINITION 2.10. *A function $f : (\Omega, \mathcal{F}) \to \mathbb{R}$ is said to be **simple** if there exist finitely many sets $A_j \in \mathcal{F}$, $1 \leq j \leq m$ and scalars λ_j so that*

$$f(x) = \sum_{1 \leq j \leq m} \lambda_j \chi_{A_j}(x)$$

The set of all simple functions is denotes $\mathcal{S}(\Omega, \mathcal{F})$ or $\mathcal{S}$ when the context is clear.

Observe that a simple function is measurable because it is the sum of measurable functions. Also a simple function takes on only finitely many values, λ_j, $1 \leq j \leq m$ and possibly 0. Conversely a measurable function that takes on finitely many values is simple. To see this, simply define, for each λ_j in the image of f, $A_j = f^{-1}(\{\lambda_j\})$. The set $\mathcal{S}$ is closed under addition, scalar multiplication and multiplication and therefore forms a linear algebra over $\mathbb{R}$ of functions. (Recall that a *linear algebra* is a vector space with a multiplication operation $(f, g) \mapsto fg$ that is associative and distributes over addition from the left and right and, for any scalar α, $\alpha(fg) = (\alpha f)g = f(\alpha g)$. Reference: Naimark's *Normed Algebras* §7.) It is further worth noting that the representation $f(x) = \sum_{1\le j\le m} \lambda_j \chi_{A_j}(x)$ in the definition of $f \in \mathcal{S}$ is *not* unique.

PROPOSITION 2.11. *Let f_n, $n \in \mathbb{N}$ be a sequence of Borel measurable real valued functions defined on a measurable space $(\Omega, \mathcal{F})$. Suppose that*

$$\lim_{n\in\mathbb{N}} f_n(x) = f(x) \quad \textit{exists in } \mathbb{R} \textit{ for every } x \in \Omega.$$

Then the function $f : (\Omega, \mathcal{F}) \to \mathbb{R}$, so defined, is measurable.

This result says, briefly, that the pointwise limit of measurable functions is measurable.

Proof: We will show that

$$f^{-1}((-\infty, a]) = \cap_{p\in\mathbb{N}} \cup_{n\in\mathbb{N}} \cap_{j\ge n} f_j^{-1}((-\infty, a + 1/p]) \tag{2.2.2}$$

for every $a \in \mathbb{R}$ The set of all sets of the form $(-\infty, a]$ generate the Borel σ-algebra $\mathcal{B}$ because any open set is generated. Therefore, we will have shown f is measurable by Proposition 2.2.

To verify (2.2.2), let $x \in f^{-1}((-\infty, a])$ so that $f(x) \le a$. Then, for any $p \in \mathbb{N}$ there is n so that $f_j(x) \le a + 1/p$ for all $j \ge n$: $x \in f_j^{-1}((-\infty, a + 1/p])$ for all $j \ge n$ or $x \in \cap_{j\ge n} f_j^{-1}((-\infty, a + 1/p])$. But p was arbitrary and so x belongs to the right side of (2.2.2) and this shows that $f^{-1}((-\infty, a])$ is a subset or equal to the right side of (2.2.2).

Conversely suppose that x belongs to the right hand side of (2.2.2). Then, for every $p \in \mathbb{N}$, there exists $n \in \mathbb{N}$ so that $f_j(x) \le a + 1/p$ for all $j \ge n$. Taking limits in this last expression as $j \to \infty$ we see that $f(x) \le a + 1/p$. Since p is arbitrary $f(x) \le a$ and this shows the right side of (2.2.2) is in $f^{-1}((-\infty, a])$ which verifies (2.2.2). This proves that f is measurable. □.

The Extended Reals $\overline{\mathbb{R}}$: We introduce the extended real line $\overline{\mathbb{R}} = [-\infty, \infty]$, sometimes referred to as the two point compactification of the real line, and this is just $\mathbb{R}$ with two points $\pm\infty$ adjoined: $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$. This will be a convenience when discussing convergence. We introduce the following topology on $\overline{\mathbb{R}}$. The neighborhoods of ∞ (resp. $-\infty$) are those sets which contain an interval of the form $(a, \infty]$

for some $a \in \mathbb{R}$ (resp. $[\infty, a)$) and the neighborhoods of $x \in \mathbb{R}$ are the usual: those sets that contain an interval of the form $\{y : |y - x| < \delta\}$ for some $\delta > 0$. Of course the topology that $\mathbb{R}$ inherits as a subset of $\overline{\mathbb{R}}$ is its usual topology. As a consequence of these definitions we see that $\overline{\mathbb{R}}$ is homeomorphic to the compact interval $[-1, 1]$ with the (usual) topology it inherits as a subset of $\mathbb{R}$. Indeed

$$\Phi(x) = \begin{cases} \frac{x}{\sqrt{x^2+1}} & \text{if} \quad x \in \mathbb{R} \\ 1 & \text{if} \quad x = \infty \\ -1 & \text{if} \quad x = -\infty \end{cases} \tag{2.2.3}$$

is continuous from $\overline{\mathbb{R}}$ to $[-1, 1]$ with inverse

$$\Phi^{-1}(x) = \begin{cases} \frac{x}{\sqrt{1-x^2}} & \text{if} \quad -1 < x < 1 \\ \infty & \text{if} \quad x = 1 \\ -\infty & \text{if} \quad x = -1 \end{cases}$$

which is also continuous.

COROLLARY 2.12. *Let f_n, $n \in \mathbb{N}$ be a sequence of Borel measurable extended real valued functions defined on a measurable space $(\Omega, \mathcal{F})$. Suppose that*

$$\lim_{n \in \mathbb{N}} f_n(x) = f(x) \quad \textit{exists in } \overline{\mathbb{R}} \textit{ for every } x \in \Omega.$$

Then the function $f : (\Omega, \mathcal{F}) \to \overline{\mathbb{R}}$, so defined, is measurable.

PROOF. An extended real valued function g is Borel measurable if and only if $\Phi \circ g$ is Borel measurable for Φ as in (2.2.3) because Φ and Φ^{-1} are continuous and hence measurable. Therefore the previous Proposition 2.11 applied to $\Phi \circ f_n$ implies implies the present result □

Measurable functions can be written as the pointwise limit of simple functions. We begin by considering nonnegative bounded functions and later we will extend to all measurable functions.

PROPOSITION 2.13. *Suppose f is be a nonnegative, bounded, measurable function defined on a measurable space $(\Omega, \mathcal{F})$. Then there is a sequence $(f_n)_{n \in \mathbb{N}}$ of simple functions such that*

1. *$(f_n)_{n \in \mathbb{N}}$ is increasing.*
2. *$0 \leq f_n \leq f$, for every $n \in \mathbb{N}$*
3. *$(f_n)_{n \in \mathbb{N}}$ converges to f pointwise and even uniformly.*

PROOF. Let $\epsilon > 0$ be given. We shall construct a simple function g_ϵ so that $g_\epsilon \leq f$ and $|f(x) - g_\epsilon(x)| < \epsilon$ for all $x \in \Omega$. Since f is bounded we have $0 \leq f \leq M$ for some positive constant M. We choose $a_0 = 0 < a_1 < a_2 < \ldots < a_m < a_{m+1}]$ so that $a_{j+1} - a_j < \epsilon$, for $0 \leq j \leq m+1$ and $M < a_{m+1} < M + \epsilon$ We define

$$A_j = \{x \in \Omega : a_j \leq f(x) < a_{j+1}\} = f^{-1}([a_j, a_{j+1}))$$

for $0 \leq j \leq m$. Then A_j is measurable and $A_j \cap A_k = \emptyset$ if $j \neq k$ and $\cup_{0 \leq j \leq m} A_j = \Omega$. We define

$$g_\epsilon(x) = \sum_{0 \leq j \leq m} a_j \chi_{A_j}$$

Then $g_\epsilon \leq f < g_\epsilon + \epsilon$, in fact, for any $x \in \Omega$, $x \in A_j$ for some j and so $g_\epsilon(x) = a_j$ and $a_j \leq f(x) < a_{j+1}$.

Define $f_1 = g_1$, $f_2 = \max\{f_1, g_{1/2}\}$, $\ldots$ $f_n = \max\{f_{n-1}, g_{1/n}\}$ It follows that $f_n \leq f < f_n + 1/n$ Also that f_n is measurable as the maximum of two simple functions. And $(f_n)_{n \in \mathbb{N}}$ is increasing and so f_n has the required properties. □

Next consider the case that f is not necessarily nonnegative but is still bounded.

PROPOSITION 2.14. *Suppose f is a real valued, bounded, measurable function defined on a measurable space $(\Omega, \mathcal{F})$. Then there is a sequence $(f_n)_{n \in \mathbb{N}}$ of simple functions such that*

1. *$|f_n| \leq |f|$, for every $n \in \mathbb{N}$*
2. *$(f_n)_{n \in \mathbb{N}}$ converges to f pointwise and even uniformly.*

Remark: The proof is based on the observation that any real valued, measurable function $f = f^+ - f^-$ where $f^+ = \max\{f, 0\} = (|f| + f)/2$ is nonnegative and measurable and $f^- = -\min\{f, 0\} = \max\{-f, 0\}$ is also nonnegative and measurable.

Proof of the Proposition 2.14 : Let g_n be a sequence of nonnegative simple functions convergent to $f^+ = \max\{f, 0\}$ as guaranteed by the preceding Proposition. Similarly let h_n be simple functions convergent to f^- and define $f_n = g_n - h_n$. Then $|f_n| = |g_n - h_n| \leq g_n + h_n \leq f^+ + f^- = |f|$. Moreover $|f - f_n| = |f^+ - g_n - (f^- - h_n)| \leq |f^+ - g_n| + |f^- - h_n|$ and since $f^+ - g_n$ and $f^- - h_n$ go to zero uniformly on Ω, f_n converges uniformly to f. □

Finally we consider the general case when f need not be bounded.

THEOREM 2.15. *Let f be a nonnegative extended value measurable function, $f : (\Omega, \mathcal{F}) \to \overline{\mathbb{R}}_+$. Then there is an increasing sequence f_n of simple functions (measurable on $(\Omega, \mathcal{F})$) so that*

1. *$0 \leq f_n \leq f$, for every $n \in \mathbb{N}$*
2. *$(f_n)_{n \in \mathbb{N}}$ converges to f pointwise.*

THEOREM 2.16. *Let f be a measurable function, $f : (\Omega, \mathcal{F}) \to \overline{\mathbb{R}}$. Then there is a sequence f_n of simple functions so that*

1. $|f_n| \leq |f|$, *for every* $n \in \mathbb{N}$
2. $(f_n)_{n\in\mathbb{N}}$ *converges to* f *pointwise.*

Remark: It is not true in general that the convergence is uniform in this more general setting. Indeed, if f is unbounded then it cannot be approximated uniformly by bounded functions. For suppose f_n be a sequence of functions, uniformly convergent to a function f and suppose each f_n is bounded (with bound depending on n). If $|f - f_n| < 1$ then $||f| - |f_n|| < 1$ which says that $|f|$ is bounded. In the setting of the Theorem above, simple functions are bounded and so we cannot expect uniform convergence.

Proof of Theorem 2.15: For each $p \in \mathbb{N}$, define $h_p = \min\{f, p\}$ so that h_p is a bounded, nonnegative measurable function and so it is possible to choose a simple function $g_p \geq 0$ so that $h_p - \frac{1}{p} \leq g_p \leq h_p$ by Proposition 2.13. Then we define $f_1 = g_1$, $f_2 = \max\{f_1, g_2\}$, ... $f_p = \max\{f_{p-1}, g_p\}$ so that f_p is simple, nonegative and increasing. We also see by induction on p, $f_p \leq h_p \leq f$. To check the pointwise convergence, suppose $x_0 \in \Omega$. If $f(x_0) < \infty$ then we may choose $p_0 \in \mathbb{N}$ so that $p_0 \geq f(x_0)$. Then for $p \geq p_0$, $f(x_0) = h_p(x_0)$ so that $f(x_0) - \frac{1}{p} = h_p(x_0) - \frac{1}{p} \leq g_p \leq f_p(x_0) \leq f(x_0)$ and so $f(x_0) = \lim_{p\in\mathbb{N}} f_p(x_0)$. On the other hand, if $f(x_0) = \infty$ then $h_p(x_0) = p$ and $g_p(x_0) \geq p - (1/p)$ so that $f_p(x_0) \geq g_p(x_0) \geq p - (1/p)$. Therefore $\lim_p f_p(x_0) = \infty = f(x_0)$. □

Proof of Theorem 2.16 The proof of Proposition 2.14 applies here except the uniform convergence there must be replaced by pointwise convergence. Let g_n be a sequence of nonnegative simple functions convergent to $f^+ = \max\{f, 0\}$ as guaranteed by the preceding Theorem. Similarly let h_n be simple functions convergent to f^- and define $f_n = g_n - h_n$. Then $|f_n| = |g_n - h_n| \leq g_n + h_n \leq f^+ + f^- = |f|$. Moreover, if $f(x) \geq 0$ then $h_n(x) = 0$ and

$$\lim_{n\in\mathbb{N}} f_n(x) = \lim_{n\in\mathbb{N}} g_n(x) = f^+(x) = f(x)$$

Similarly $f(x) < 0$, then $g_n(x) = 0$ and

$$\lim_{n\in\mathbb{N}} f_n(x) = \lim_{n\in\mathbb{N}} -h_n(x) = -f^-(x) = f(x)$$

Note that $\lim_n g_n(x) = \infty$ and $\lim_n h_n(y) = \infty$ are both possible, but only if $x \neq y$. □

The set of simple functions is an algebra as we have seen. Moreover if $\phi : \mathbb{R} \to \mathbb{R}$ is Borel measurable then $\phi \circ f$ is simple whenever f is.

In particular $|f|$ is simple if f is and $\max\{f, g\} = \frac{1}{2}(f + g + |f - g|)$ is simple whenever f and g are. Similarly $\min\{f, g\}$ is simple.