Assignment 8, Math 4820\5820
Due November 30

1. Suppose that $p$ is a point and $p_n$ is a sequence with the property that every subsequence of $p_n$ has a sub-subsequence that converges to $p$. (The limit $p$ does not depend on the subsequence.) Show that the sequence $p_n$ itself converges to $p$.

2. Define $s_1 = \sqrt{2}$ and, for each $n \in \mathbb{N}$ define $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$ Show that $s_n \leq 2$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n$ exists.

3. Suppose that $a_n$, $b_n$ are two sequences that are bounded above. (a) Show that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

and (b) give an example to show that strict inquality may hold: that is

$$\limsup_{n\to\infty}(a_n + b_n) < \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

is possible.

4. Consider $\lim_{n\to\infty} x^n$ when $x$ is real. Does the limit exist? When is it $\pm\infty$? You may need to consider cases. Justify your conclusions.

5. Let $P$ be the "middle thirds" Cantor set. Show that $x \in [0, 1]$ is in $P$ if and only if $x$ has a ternary expansion (base 3) which consists solely of zeroes and twos. Suggestion. Show that every $x$ is the removed middle third (1/3,2/3) has ternary expansion beginning with a one. Also remember that $[1/3]_3 = 0.022222222\ldots$.