

Assignment 6, Math 4820\5820

Due Friday, October 28

1. Let $H = \{1/n : n \in \mathbb{N}\}$.
 - (a) Construct a sequence of open sets G_n in \mathbb{R} so that $H \subseteq \cup_{n=1}^{\infty} G_n$ but there is no m so that $H \subseteq \cup_{n=1}^m G_n$
 - (b) Show that if $G_n, n \in \mathbb{N}$ is any sequence of open sets in \mathbb{R} so that $H \cup \{0\} \subseteq \cup_n^{\infty} G_n$ then there is $m \in \mathbb{N}$ so that $H \cup \{0\} \subseteq \cup_{n=1}^m G_n$. (Try to avoid using the Heine Borel Theorem.)
2. Recall the inequality: $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$. Determine when the inequality becomes equality. In other words find conditions on $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ that are equivalent to $|\mathbf{x} \cdot \mathbf{y}| = |\mathbf{x}||\mathbf{y}|$. (You will need to show necessity and sufficiency.)
3. Suppose that X is a metric space with metric $d : X \times X \rightarrow \mathbb{R}$. Show that

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

also defines a metric on X . Suggestion: Show that $f(t) = t/(1 + t)$ is an increasing function on $[0, \infty)$. Observe that $d_1(x, y) \leq 1$ for all $x, y \in X$

4. For X, d and d_1 as in the previous problem show that a set is open in the metric space (X, d) if and only if it is open in the metric space (X, d_1) .