

- 7.18 Prove parts (a) and (b) of Theorem 7.17.
- 7.19 Prove Theorem 7.19(b). That is, suppose that $f: A \rightarrow B$ and $g: B \rightarrow C$ are both injective. Prove that $g \circ f: A \rightarrow C$ is injective.
- 7.20 Suppose that $f: A \rightarrow B$ and suppose that $C \subseteq A$ and $D \subseteq B$.
- Prove or give a counterexample: $f(C) \subseteq D$ iff $C \subseteq f^{-1}(D)$.
 - What condition on f will ensure that $f(C) = D$ iff $C = f^{-1}(D)$? Prove your answer.
- 7.21 Suppose that $f: A \rightarrow B$ and let C be a subset of A .
- Prove or give a counterexample: $f(A \setminus C) \subseteq f(A) \setminus f(C)$.
 - Prove or give a counterexample: $f(A) \setminus f(C) \subseteq f(A \setminus C)$.
 - What condition on f will ensure that $f(A \setminus C) = f(A) \setminus f(C)$? Prove your answer. ☆
 - What condition on f will ensure that $f(A \setminus C) = B \setminus f(C)$? Prove your answer.
- 7.22 Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Define the functions
 $f = \{(1, a), (2, b), (3, a)\}$ and $g = \{(a, 1), (b, 3), (c, 2)\}$.
 Describe each of the following functions by listing its ordered pairs. Then state its range.
- g^{-1}
 - $f \circ g$
 - $g \circ f$
 - $f \circ g \circ f$
- 7.23 Let f and g be functions between A and B . Prove that $f = g$ iff $\text{dom } f = \text{dom } g$ and for every $x \in \text{dom } f$, $f(x) = g(x)$. ☆
- 7.24 Suppose that $f: A \rightarrow B$, $g: B \rightarrow C$, and $h: C \rightarrow D$. Prove that $h \circ (g \circ f) = (h \circ g) \circ f$.
- 7.25 Let $f: A \rightarrow B$ and $g: B \rightarrow C$. Using the ordered pair definition of the composition $g \circ f$, prove that $g \circ f$ is a function and that $g \circ f: A \rightarrow C$. ☆
- 7.26 Find an example of functions $f: A \rightarrow B$ and $g: B \rightarrow C$ such that f and $g \circ f$ are both injective, but g is not injective.
- 7.27 Find an example of functions $f: A \rightarrow B$ and $g: B \rightarrow C$ such that g and $g \circ f$ are both surjective, but f is not surjective.
- 7.28 Find an example of functions $f: A \rightarrow B$ and $g: B \rightarrow C$ such that $g \circ f$ is bijective, but neither f nor g is bijective.
- 7.29 Let $f: A \rightarrow B$ and suppose that there exists a function $g: B \rightarrow A$ such that $g \circ f = i_A$ and $f \circ g = i_B$.
- Prove that f is bijective.
 - Prove that $g = f^{-1}$. ☆
- 7.30 Suppose that $g: A \rightarrow C$ and $h: B \rightarrow C$. Prove that if h is bijective, then there exists a function $f: A \rightarrow B$ such that $g = h \circ f$. *Hint:* Draw a picture.

surjective function and an injective function.

- Let A be the set of all students in the school. Define $f: A \rightarrow [0, 200]$ by " $f(x)$ is the age of x ." Describe the functions h and g as given above.

Section 8 CARDINALITY

How can we compare the sizes of two sets? If $S = \{x \in \mathbb{R} : x^2 = 9\}$, then $S = \{-3, 3\}$ and we say that S has two elements. If $T = \{1, 7, 11\}$, then T has three elements and we think of T as being "larger" than S . These intuitive ideas are fine for small (finite) sets, but how can we compare the size of (infinite) sets like \mathbb{N} or \mathbb{R} ?

We shall begin by deciding what it means for two sets to be the same size and then approach the question of comparing size. Certainly, it is reasonable

to say that two sets S and T are the same size if there is a bijective function $f: S \rightarrow T$, for this function will set up a one-to-one correspondence between the elements of each set.

8.1 DEFINITION Two sets S and T are called **equinumerous**, and we write $S \sim T$, if there exists a bijective function from S onto T .[†]

8.2 PRACTICE If \mathcal{F} is a family of sets, then the concept of being equinumerous is a relation on \mathcal{F} . Show that “ \sim ” is an equivalence relation in the sense of Definition 6.9.

Cardinal Numbers

Since “ \sim ” is an equivalence relation, it partitions any family of sets into disjoint equivalence classes. With each equivalence class we associate a cardinal number that we think of as giving the size of the set. Technically, a cardinal number is sometimes defined to be an equivalence class determined by the relation \sim . But this raises questions about the domain of \sim . That is, just what sets do we include in the family \mathcal{F} on which \sim is defined? Since we wish to avoid these complications, it will be simpler merely to associate a cardinal number with each equivalence class. This leaves the question “What is a cardinal number?” unanswered, but it will be adequate for our purposes. Given any two sets, it makes sense to ask if they have the same cardinal number. They will if and only if they are equinumerous.

Using the concept of two sets being equinumerous, we can classify sets according to size.

8.3 DEFINITION A set S is said to be **finite** if $S = \emptyset$ or if there exists $n \in \mathbb{N}$ and a bijection $f: \{1, 2, \dots, n\} \rightarrow S$. If a set is not finite, it is said to be **infinite**.

It will be convenient to abbreviate the set $\{1, 2, \dots, n\}$ by I_n . Thus we can say that S is finite iff $S = \emptyset$ or S is equinumerous with I_n for some $n \in \mathbb{N}$.

8.4 DEFINITION The **cardinal number** of I_n is n , and if $S \sim I_n$, we say that S has n elements. The cardinal number of \emptyset is taken to be 0. If a cardinal number is not finite, it is called **transfinite**.

8.5 PRACTICE Suppose S and T are both sets having n elements. Then from Definition 8.3 there exist bijections $f: I_n \rightarrow S$ and $g: I_n \rightarrow T$. Show that S and T are equinumerous directly by finding a bijection $h: S \rightarrow T$.

[†] Some authors use “equivalent” or “set equivalent” instead of “equinumerous.”

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8.6 DEFINITION A set S is said to be **denumerable** if there exists a bijection $f: \mathbb{N} \rightarrow S$. If a set is finite or denumerable, it is called **countable**. If a set is not countable, it is **uncountable**. The cardinal number of a denumerable set is denoted by \aleph_0 .[†]

In other words, a set is denumerable iff it is equinumerous with the set of natural numbers \mathbb{N} . The relationships between the various “sizes” of sets are illustrated in Figure 8.1.

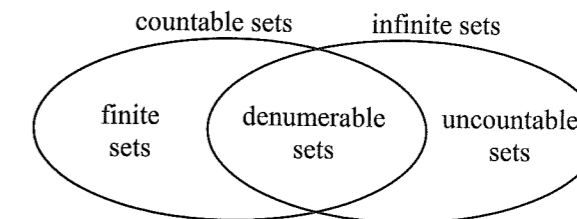


Figure 8.1

Since the set \mathbb{N} is not finite, there exists at least one infinite set.[‡] It is not the case, however, that every infinite set has \aleph_0 as its cardinal number. That is, not every infinite set is denumerable. It will turn out that there are many different sizes of infinity. Before showing this surprising fact, let us look carefully at some of the properties of countable sets.

Countable Sets

8.7 EXAMPLE It would seem at first glance that the set \mathbb{N} of natural numbers should be “bigger” than the set E of even natural numbers. Indeed, E is a proper subset of \mathbb{N} and, in fact, it contains only “half” of \mathbb{N} . But what is “half” of \aleph_0 ? Our experience with finite sets is a poor guide here, for \mathbb{N} and E are actually equinumerous! The function $f(n) = 2n$ is a bijection from \mathbb{N} onto E , so E also has cardinality \aleph_0 .

[†] The symbol \aleph_0 is read “aleph naught” or “aleph zero.” Aleph is the first letter of the Hebrew alphabet.

[‡] The “obvious” fact that \mathbb{N} is infinite is actually nontrivial to prove. The proof is based on the Dirichlet pigeonhole principle: If $n > m$, then there exists no injection $f: I_n \rightarrow I_m$. That is, if n pigeons must fit into m pigeonholes with $n > m$, then at least two pigeons will end up in the same hole. The pigeonhole principle in turn depends on the principle of mathematical induction (Theorem 10.2). See Henkin and others (1962), page 125, for a complete proof.

8.8 PRACTICE Find a bijection $f: \mathbb{N} \rightarrow \mathbb{Z}$, thereby showing that the set \mathbb{Z} of all integers is also denumerable.

If a nonempty set S is finite, then there exist $n \in \mathbb{N}$ and a bijection $f: I_n \rightarrow S$. Using the function f , we can count off the members of S as follows: $f(1), f(2), f(3), \dots, f(n)$. Letting $f(k) = s_k$ for $1 \leq k \leq n$, we obtain the more familiar notation $S = \{s_1, s_2, \dots, s_n\}$. The same kind of counting process is possible for a denumerable set, and this is why both kinds of sets are called countable. For example, if T is denumerable, then there exists a bijection $g: \mathbb{N} \rightarrow T$, and we may write $T = \{g(1), g(2), g(3), \dots\}$ or $T = \{t_1, t_2, t_3, \dots\}$, where $g(n) = t_n$.

This ability to list the members of a set as a first, second, third and so on, characterizes countable sets. If the list terminates, then the set is finite. On the other hand, if t_1, t_2, t_3, \dots is a nonterminating list of the members of T without repetitions, then T is denumerable since the function $g: \mathbb{N} \rightarrow T$ given by $g(n) = t_n$ will be bijective.

8.9 THEOREM Let S be a countable set and let $T \subseteq S$. Then T is countable.

Proof: If T is finite, then we are done. Thus we may assume that T is infinite. This implies (Exercise 8.6) that S is infinite, so S is denumerable (since it is countable). Therefore, there exists a bijection $f: \mathbb{N} \rightarrow S$ and we can write S as a list of distinct members

$$S = \{s_1, s_2, s_3, \dots\}$$

where $f(n) = s_n$. Now let

$$A = \{n \in \mathbb{N} : s_n \in T\}.$$

Since A is a nonempty subset of \mathbb{N} , the Well-Ordering Property of \mathbb{N} (Axiom 10.1) implies A has a least member, say a_1 . Similarly, the set $A \setminus \{a_1\}$ has a least member, say a_2 . In general, having chosen a_1, \dots, a_k , let a_{k+1} be the least member in $A \setminus \{a_1, \dots, a_k\}$. Essentially, if we select from our listing of S those terms that are in T and keep them in the same order, then a_n is the subscript of the n th term in this new list.

Now define a function $g: \mathbb{N} \rightarrow \mathbb{N}$ by $g(n) = a_n$. Since T is infinite, g is defined for every $n \in \mathbb{N}$. Since $a_{n+1} \notin \{a_1, \dots, a_n\}$, g must be injective. Thus the composition $f \circ g$ is also injective. Since each element of T is somewhere in the listing of S , $g(\mathbb{N})$ includes all the subscripts of terms in T . Thus $f \circ g$ is a bijection from \mathbb{N} onto T and T is denumerable. ♦

Using Theorem 8.9, we can derive two very useful criteria for determining when a set is countable.

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Using Theorem 8.9, we can derive two very useful criteria for determining when a set is countable.

8.10 THEOREM Let S be a nonempty set. The following three conditions are equivalent.

- (a) S is countable.
- (b) There exists an injection $f: S \rightarrow \mathbb{N}$.
- (c) There exists a surjection $g: \mathbb{N} \rightarrow S$.

Proof: Suppose that S is countable. Then there exists a bijection $h: J \rightarrow S$, where $J = I_n$ for some $n \in \mathbb{N}$ if S is finite and $J = \mathbb{N}$ if S is denumerable. In either case, h^{-1} is a bijection from S onto J and hence an injection (at least) from S to \mathbb{N} . Thus (a) implies (b).

Now suppose that there exists an injection $f: S \rightarrow \mathbb{N}$. Then f is a bijection from S to $f(S)$, so f^{-1} is a bijection from $f(S)$ back to S . We use f^{-1} to obtain a function g from all of \mathbb{N} onto S as follows: Let p be any fixed member of S . Define $g: \mathbb{N} \rightarrow S$ by

$$g(n) = \begin{cases} f^{-1}(n), & \text{if } n \in f(S) \\ p, & \text{if } n \notin f(S). \end{cases}$$

Then $g[f(S)] = f^{-1}[f(S)] = S$ and $g[\mathbb{N} \setminus f(S)] = \{p\}$, so that g is a surjection from \mathbb{N} onto S . Thus (b) implies (c).

Finally, suppose that there exists a surjection $g: \mathbb{N} \rightarrow S$. Define $h: S \rightarrow \mathbb{N}$ by

$$h(s) \text{ is the smallest } n \in \mathbb{N} \text{ such that } g(n) = s.$$

Then h is an injection from S to \mathbb{N} , and hence a bijection from S onto the subset $h(S)$ of \mathbb{N} . Since \mathbb{N} is countable, Theorem 8.9 implies that $h(S)$ is countable. Since S and $h(S)$ are equinumerous, S is also countable. ♦

8.11 EXAMPLE

(a) Let S and T be nonempty countable sets. We shall show that $S \cup T$ is countable. Since S and T are countable, Theorem 8.10 implies that there exist surjections $f: \mathbb{N} \rightarrow S$ and $g: \mathbb{N} \rightarrow T$. Define $h: \mathbb{N} \rightarrow S \cup T$ by

$$h(n) = \begin{cases} f\left(\frac{n+1}{2}\right), & \text{if } n \text{ is odd} \\ g\left(\frac{n}{2}\right), & \text{if } n \text{ is even.} \end{cases}$$

(See Figure 8.2.) Then h is surjective, so $S \cup T$ is countable. Notice how the use of Theorem 8.10 allowed us to consider both the finite and denumerable cases at the same time. It also meant that we did not need to assume that S and T are disjoint. [If $S \cap T \neq \emptyset$, then $f(m) = g(n)$ for some $m \neq n$. In this case h will not be injective, even if both f and g are.]

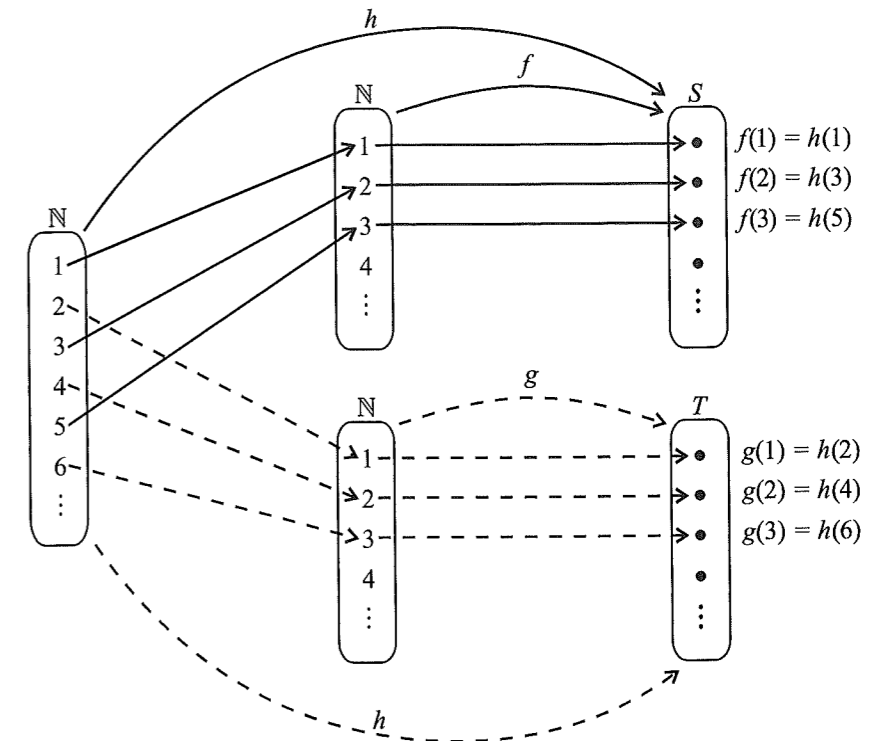


Figure 8.2

(b) Recall that every natural number can be written as a product of primes, and this representation is unique except for the order of the factors. For example, $12 = 2^2 \cdot 3$. Using this fact and Theorem 8.10, we can show that the Cartesian product of two countable sets is countable. Suppose that S and T are nonempty countable sets. Then there exist injections $f: S \rightarrow \mathbb{N}$ and $g: T \rightarrow \mathbb{N}$. Define $h: S \times T \rightarrow \mathbb{N}$ by

$$h(s, t) = 2^{f(s)} \cdot 3^{g(t)}, \quad \text{where } s \in S \text{ and } t \in T.$$

Then h is injective, for if $h(s, t) = h(u, v)$, then

$$2^{f(s)} \cdot 3^{g(t)} = 2^{f(u)} \cdot 3^{g(v)}.$$

Since the prime factored form of a number is unique, we have $f(s) = f(u)$ and $g(t) = g(v)$. Finally, since f and g are injective, this implies that $s = u$ and $t = v$. Thus, since h is injective, we conclude (by Theorem 8.10 again) that $S \times T$ is countable.

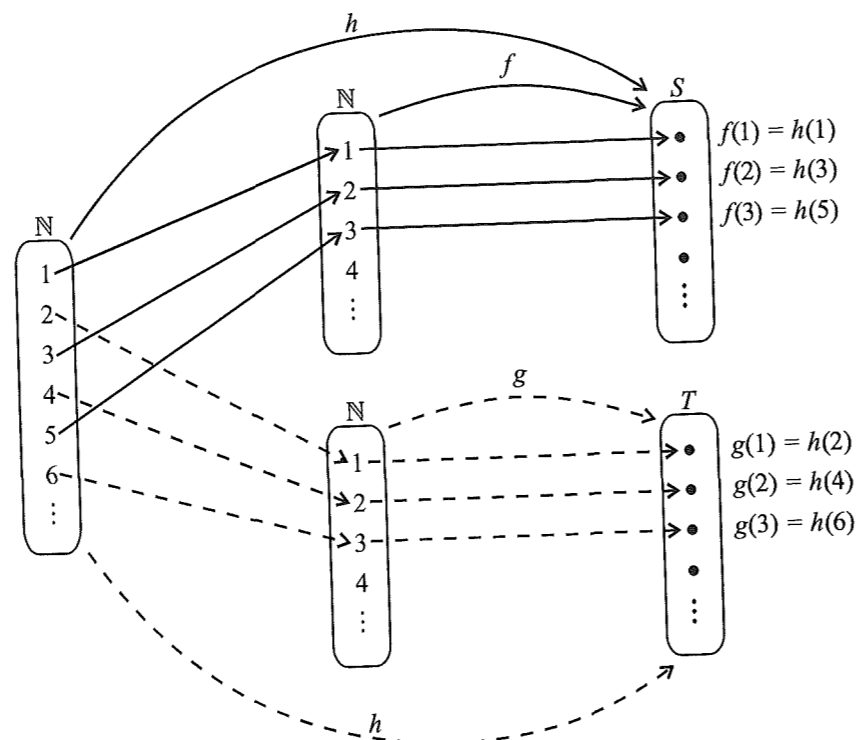


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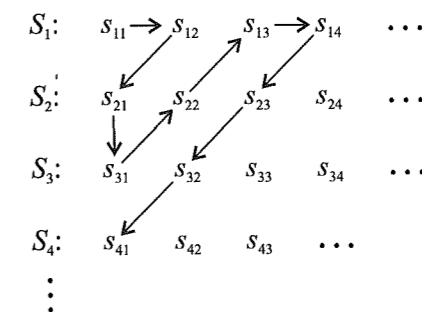
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(c) Following the approach used in the preceding example, we can show that the set \mathbb{Q} of rational numbers is countable. To begin with, let \mathbb{Q}^+ and \mathbb{Q}^- be the set of positive rationals and negative rationals, respectively. We first show that \mathbb{Q}^+ is countable. Any member of \mathbb{Q}^+ can be written uniquely as m/n , where $m, n \in \mathbb{N}$, $n \neq 0$, and m and n are relatively prime (have no common prime divisors). Define $f: \mathbb{Q}^+ \rightarrow \mathbb{N}$ by

$$f(m/n) = 2^m \cdot 3^n.$$

Then f is injective as in the previous example, so \mathbb{Q}^+ is countable by Theorem 8.10. The mapping $g: \mathbb{Q}^+ \rightarrow \mathbb{Q}^-$ given by $g(r) = -r$ is clearly bijective, so \mathbb{Q}^+ and \mathbb{Q}^- are equinumerous. That is, \mathbb{Q}^- is countable. Since $\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+$, by applying part (a) twice we see that \mathbb{Q} is countable.

(d) By generalizing on the approach used in part (a), we can show that the union of a countable family of countable sets is countable. To see this, let $\{S_\alpha: \alpha \in \mathcal{A}\}$ be such a family. Since empty sets contribute nothing to the union, we may assume that all the sets are nonempty. Since the family is countable, we can replace the index set by \mathbb{N} and consider $\{S_n: n \in \mathbb{N}\}$. If the original family had only a finite number of sets, S_1, \dots, S_k , let $S_n = S_1$ for all $n > k$. Now for each set S_n there exists a surjection $f_n: \mathbb{N} \rightarrow S_n$, so we can write $S_n = \{s_{n1}, s_{n2}, s_{n3}, \dots\}$, where $f_n(j) = s_{nj}$. We now arrange the elements of $\bigcup_{n=1}^{\infty} S_n$ in a rectangular array:



By moving along each diagonal of the array in the manner indicated, we obtain a listing of all the elements in $\bigcup_{n=1}^{\infty} S_n$:

$$s_{11}, s_{12}, s_{21}, s_{31}, s_{22}, s_{13}, s_{14}, \dots$$

This listing defines a surjection $f: \mathbb{N} \rightarrow \bigcup_{n=1}^{\infty} S_n$, so that the union is countable.[†]

Having seen several examples of countable sets, the reader may be wondering what sets (if any) are uncountable. We now show that the set of

[†] Our argument has used the axiom of choice in a subtle way. See Section 9.

all real numbers is uncountable. Our proof uses the “diagonal process” developed by Georg Cantor in the late nineteenth century.

8.12 THEOREM The set \mathbb{R} of real numbers is uncountable.

Proof: Since any subset of a countable set is countable (Theorem 8.9), it suffices to show that the interval $J = (0, 1)$ is uncountable. If J were countable, we could list its members and have

$$J = \{x_1, x_2, x_3, \dots\} = \{x_n : n \in \mathbb{N}\}.$$

We shall show that this leads to a contradiction by constructing a real number that is in J but is not included in the list of x_n 's. Each element of J has an infinite decimal expansion, so we can write

$$\begin{aligned} x_1 &= 0.a_{11}a_{12}a_{13}\cdots, \\ x_2 &= 0.a_{21}a_{22}a_{23}\cdots, \\ x_3 &= 0.a_{31}a_{32}a_{33}\cdots, \\ &\vdots \end{aligned}$$

where each $a_{ij} \in \{0, 1, \dots, 9\}$. (Some numbers, such as $0.5000\cdots = 0.4999\cdots$, have more than one representation, but this will not be a problem.) We now construct a real number $y = 0.b_1b_2b_3\cdots$ by defining

$$b_n = \begin{cases} 2, & \text{if } a_{nn} \neq 2, \\ 3, & \text{if } a_{nn} = 2. \end{cases}$$

Since each digit in the decimal expansion of y is either 2 or 3, $y \in J$. But y is not one of the numbers x_n , since it differs from x_n in the n th decimal place. (Since none of the digits in y are 0 or 9, it is not one of the numbers with two representations.) This contradicts our assumption that J is countable, so J must be uncountable. ♦

8.13 PRACTICE Show that the set of irrational numbers is uncountable.

Ordering of Cardinals[†]

We conclude this section by returning to our original question about comparing the size of two sets. We would like to make some sense out of the notion that one set is “bigger” or has “more” points than another set. For finite sets we observe that, if S is a proper subset of T , then T certainly has “more” points than S . Unfortunately, this does not hold for infinite sets, as we saw in Example 8.7. In fact, it can be shown (Exercise 8.12) that any

[†] The remainder of this section may be omitted on a first reading.

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We shall show that this leads to a contradiction by constructing a real number that is in J but is not included in the list of x_n 's. Each element of J has an infinite decimal expansion, so we can write

$$\begin{aligned} x_1 &= 0.a_{11}a_{12}a_{13}\cdots, \\ x_2 &= 0.a_{21}a_{22}a_{23}\cdots, \\ x_3 &= 0.a_{31}a_{32}a_{33}\cdots, \\ &\vdots \end{aligned}$$

where each $a_{ij} \in \{0, 1, \dots, 9\}$. (Some numbers, such as $0.5000\cdots = 0.4999\cdots$, have more than one representation, but this will not be a problem.) We now construct a real number $y = 0.b_1b_2b_3\cdots$ by defining

$$b_n = \begin{cases} 2, & \text{if } a_{nn} \neq 2, \\ 3, & \text{if } a_{nn} = 2. \end{cases}$$

Since each digit in the decimal expansion of y is either 2 or 3, $y \in J$. But y is not one of the numbers x_n , since it differs from x_n in the n th decimal place. (Since none of the digits in y are 0 or 9, it is not one of the numbers with two representations.) This contradicts our assumption that J is countable, so J must be uncountable. ♦

8.13 PRACTICE Show that the set of irrational numbers is uncountable.

Ordering of Cardinals[†]

We conclude this section by returning to our original question about comparing the size of two sets. We would like to make some sense out of the notion that one set is “bigger” or has “more” points than another set. For finite sets we observe that, if S is a proper subset of T , then T certainly has “more” points than S . Unfortunately, this does not hold for infinite sets, as we saw in Example 8.7. In fact, it can be shown (Exercise 8.12) that any

[†] The remainder of this section may be omitted on a first reading.

infinite set is equinumerous with a proper subset of itself. Thus the property of being a proper subset is not an adequate basis for comparing the size of sets in general.

A more fruitful approach is built on our definition of equinumerous and our understanding of functions. Intuitively, if $f: S \rightarrow T$ is injective, then S can be no larger than T . Not only is this true for finite sets, but we have observed that it also holds for countable sets (Theorem 8.10). Since we think of cardinal numbers as representing the size of a set, we shall use them when comparing sizes.

8.14 DEFINITION We denote the cardinal number of a set S by $|S|$, so that we have $|S| = |T|$ iff S and T are equinumerous. That is, $|S| = |T|$ iff there exists a bijection $f: S \rightarrow T$. In light of our discussion above, we define $|S| \leq |T|$ to mean that there exists an injection $f: S \rightarrow T$. As usual, $|S| < |T|$ means that $|S| \leq |T|$ and $|S| \neq |T|$.

The basic properties of our ordering of cardinals are included in Theorem 8.15. The proofs are all straightforward and are left to the exercises. Part (a) corresponds to our intuitive feeling about the relative sizes of subsets. Parts (b) and (c) are the reflexive and transitive properties, respectively. Part (d) means that the order of m and n as integers is the same as the order for the finite cardinals m and n . (Recall that $|\{1, 2, \dots, m\}|$ is denoted by m .)

8.15 THEOREM Let S , T , and U be sets.

- If $S \subseteq T$, then $|S| \leq |T|$.
- $|S| \leq |S|$.
- If $|S| \leq |T|$ and $|T| \leq |U|$, then $|S| \leq |U|$.
- If $m, n \in \mathbb{N}$ and $m \leq n$, then $|\{1, 2, \dots, m\}| \leq |\{1, 2, \dots, n\}|$.
- If S is finite, then $|S| < \aleph_0$.

Proof: Exercise 8.8. ♦

It is customary to denote the cardinal number of \mathbb{R} by c , for continuum. Since $\mathbb{Q} \subseteq \mathbb{R}$, we have $\aleph_0 \leq c$. In fact, since \mathbb{Q} is countable and \mathbb{R} is uncountable, we have $\aleph_0 < c$. Thus Theorem 8.15(e) implies that \aleph_0 and c are unequal transfinite cardinals. Are there any others? The answer is an emphatic yes, as we see in our next theorem.

8.16 DEFINITION Given any set S , let $\mathcal{P}(S)$ denote the collection of all the subsets of S . The set $\mathcal{P}(S)$ is called the **power set** of S .

8.17 PRACTICE List all the elements of $\mathcal{P}(S)$, when $S = \{a, b, c\}$. Find $|S|$ and $|\mathcal{P}(S)|$ and note that $|S| < |\mathcal{P}(S)|$.

8.18 THEOREM For any set S , we have $|S| < |\mathcal{P}(S)|$.

Proof: The function $g: S \rightarrow \mathcal{P}(S)$ given by $g(s) = \{s\}$ is clearly injective, so $|S| \leq |\mathcal{P}(S)|$. To prove that $|S| \neq |\mathcal{P}(S)|$, we show that no function from S to $\mathcal{P}(S)$ can be surjective. Suppose that $f: S \rightarrow \mathcal{P}(S)$. Then for each $x \in S$, $f(x)$ is a subset of S . Now for some x in S it may be that x is in the subset $f(x)$ and for others it may not be. Let

$$T = \{x \in S : x \notin f(x)\}.$$

Now $T \subseteq S$, so $T \in \mathcal{P}(S)$. If f were surjective, then $T = f(y)$ for some $y \in S$. Now either $y \in T$ or $y \notin T$, but both possibilities lead to contradictions: If $y \in T$, then $y \notin f(y)$ by the definition of T . But $f(y) = T$, so $y \notin f(y)$ implies $y \in T$. On the other hand, if $y \notin T$, then since $f(y) = T$, we have $y \in f(y)$. But then $y \in T$, by the definition of T .

Thus we conclude that no function from S to $\mathcal{P}(S)$ can be surjective, so $|S| \neq |\mathcal{P}(S)|$. ♦

By applying Theorem 8.18 again and again, we obtain an infinite sequence of transfinite cardinals each larger than the one preceding:

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < \dots$$

Does the cardinal c fit into this sequence? In Exercise 8.24 we sketch the proof that $|\mathcal{P}(\mathbb{N})| = c$. In Exercise 8.11 we show that every infinite set has a denumerable subset. Since (by Theorem 8.9) every infinite subset of a denumerable set is denumerable, we see that \aleph_0 is the smallest transfinite cardinal.

What is the first cardinal greater than \aleph_0 ? We know that $c > \aleph_0$, but is there any cardinal number λ such that

$$\aleph_0 < \lambda < c?$$

More specifically, is there any subset of \mathbb{R} with size “in between” \mathbb{N} and \mathbb{R} ? Experience tells us that there is not, because no such set has ever been found. The conjecture that there is no such set was first made by Cantor and is known as the **continuum hypothesis**. In 1900 it was included as the first of Hilbert’s famous 23 unsolved problems. Whether it is true or false is still an unanswered—perhaps unanswerable—question. It is known, however, that the assumption of the continuum hypothesis does not contradict any of the usual axioms of set theory. (This was proved by Kurt Gödel in 1938.) But lest we take too much comfort in this, we should also point out that it has been proved (by Paul Cohen in 1963) that the denial of the continuum hypothesis does not lead to any contradictions either.

Thus the continuum hypothesis is undecidable on the basis of the currently accepted axioms for set theory. (It can be neither proved nor disproved.) It remains to be seen whether new axioms will be found that will enable future mathematicians finally to settle the issue.

8.18 THEOREM For any set S , we have $|S| < |\mathcal{P}(S)|$.

Proof: The function $g: S \rightarrow \mathcal{P}(S)$ given by $g(s) = \{s\}$ is clearly injective, so $|S| \leq |\mathcal{P}(S)|$. To prove that $|S| \neq |\mathcal{P}(S)|$, we show that no function from S to $\mathcal{P}(S)$ can be surjective. Suppose that $f: S \rightarrow \mathcal{P}(S)$. Then for each $x \in S$, $f(x)$ is a subset of S . Now for some x in S it may be that x is in the subset $f(x)$ and for others it may not be. Let

$$T = \{x \in S : x \notin f(x)\}.$$

Now $T \subseteq S$, so $T \in \mathcal{P}(S)$. If f were surjective, then $T = f(y)$ for some $y \in S$. Now either $y \in T$ or $y \notin T$, but both possibilities lead to contradictions: If $y \in T$, then $y \notin f(y)$ by the definition of T . But $f(y) = T$, so $y \notin f(y)$ implies $y \in T$. On the other hand, if $y \notin T$, then since $f(y) = T$, we have $y \in f(y)$. But then $y \in T$, by the definition of T .

Thus we conclude that no function from S to $\mathcal{P}(S)$ can be surjective, so $|S| \neq |\mathcal{P}(S)|$. ♦

By applying Theorem 8.18 again and again, we obtain an infinite sequence of transfinite cardinals each larger than the one preceding:

$$\aleph_0 = |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < \dots$$

Does the cardinal c fit into this sequence? In Exercise 8.24 we sketch the proof that $|\mathcal{P}(\mathbb{N})| = c$. In Exercise 8.11 we show that every infinite set has a denumerable subset. Since (by Theorem 8.9) every infinite subset of a denumerable set is denumerable, we see that \aleph_0 is the smallest transfinite cardinal.

What is the first cardinal greater than \aleph_0 ? We know that $c > \aleph_0$, but is there any cardinal number λ such that

$$\aleph_0 < \lambda < c?$$

More specifically, is there any subset of \mathbb{R} with size “in between” \mathbb{N} and \mathbb{R} ? Experience tells us that there is not, because no such set has ever been found. The conjecture that there is no such set was first made by Cantor and is known as the **continuum hypothesis**. In 1900 it was included as the first of Hilbert’s famous 23 unsolved problems. Whether it is true or false is still an unanswered—perhaps unanswerable—question. It is known, however, that the assumption of the continuum hypothesis does not contradict any of the usual axioms of set theory. (This was proved by Kurt Gödel in 1938.) But lest we take too much comfort in this, we should also point out that it has been proved (by Paul Cohen in 1963) that the denial of the continuum hypothesis does not lead to any contradictions either.

Thus the continuum hypothesis is undecidable on the basis of the currently accepted axioms for set theory. (It can be neither proved nor disproved.) It remains to be seen whether new axioms will be found that will enable future mathematicians finally to settle the issue.

Review of Key Terms in Section 8

Equinumerous	Cardinal number	Countable
Finite	Transfinite	Uncountable
Infinite	Denumerable	Power set

ANSWERS TO PRACTICE PROBLEMS

- 8.2 The identity function is a bijection, so $S \sim S$. Thus “ \sim ” is reflexive. Now, if $S \sim T$, then there exists a bijection $f: S \rightarrow T$. By Theorem 7.23, $f^{-1}: T \rightarrow S$ is also bijective, so $T \sim S$ and “ \sim ” is symmetric. Finally, suppose that $S \sim T$ and $T \sim U$. Then there exist bijections $f: S \rightarrow T$ and $g: T \rightarrow U$. By Theorem 7.19, $g \circ f: S \rightarrow U$ is bijective, so $S \sim U$. Thus “ \sim ” is transitive.
- 8.5 Since f is a bijection from I_n onto S , f^{-1} is a bijection from S onto I_n . Thus if we follow f^{-1} by g , we get a bijection from S onto T . That is, let $h = g \circ f^{-1}$.
- 8.8 One such function is defined by $f(n) = n/2$ for n even and $f(n) = -(n-1)/2$ for n odd.
- 8.13 The rationals are countable by Example 8.11(c). If the irrationals were also countable, then \mathbb{R} (their union) would be countable by Example 8.11(a). Since \mathbb{R} is uncountable, so are the irrationals.
- 8.17 $\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$. We have $|S| = 3$ and $|\mathcal{P}(S)| = 8$, so $|S| < |\mathcal{P}(S)|$.

EXERCISES

*Exercises marked with * are used in later sections and exercises marked with ☆ have hints or solutions in the back of the book.*

- 8.1 Mark each statement True or False. Justify each answer.
 - (a) Two sets S and T are equinumerous if there exists a bijection $f: S \rightarrow T$.
 - (b) If a set S is finite, then S is equinumerous with I_n for some $n \in \mathbb{N}$.
 - (c) If a cardinal number is not finite, it is said to be infinite.
 - (d) A set S is denumerable if there exists a bijection $f: \mathbb{R} \rightarrow S$.
 - (e) Every subset of a countable set is countable.
 - (f) Every subset of a denumerable set is denumerable.
- 8.2 Mark each statement True or False. Justify each answer.
 - (a) A nonempty set S is countable iff there exists an injection $f: \mathbb{N} \rightarrow S$.
 - (b) The set \mathbb{Q} of rational numbers is denumerable.
 - (c) Let S be a nonempty set. There exists an injection $f: S \rightarrow \mathbb{N}$ iff there exists a surjection $g: \mathbb{N} \rightarrow S$.
 - (d) The set \mathbb{R} of real numbers is denumerable.

- (e) $|S| \leq |T|$ means that there exists an injection $f: S \rightarrow T$.
 (f) The continuum hypothesis says that \aleph_0 is the smallest transfinite cardinal number.

8.3 Show that the following pairs of sets S and T are equinumerous by finding a specific bijection between the sets in each pair.

- (a) $S = [0, 1]$ and $T = [1, 3]$
 (b) $S = [0, 1]$ and $T = [0, 1)$ ☆
 (c) $S = [0, 1)$ and $T = (0, 1)$
 (d) $S = (0, 1)$ and $T = (0, \infty)$
 (e) $S = (0, 1)$ and $T = \mathbb{R}$

8.4 (a) Suppose that $m < n$. Prove that the intervals $(0, 1)$ and (m, n) are equinumerous by finding a specific bijection between them.

(b) Use part (a) to prove that any two open intervals are equinumerous.

8.5 Prove: If $(S \setminus T) \sim (T \setminus S)$, then $S \sim T$. ☆

8.6 Prove: Every subset of a finite set is finite.

8.7 Use Example 8.11(d) to prove that \mathbb{Q} is countable.

8.8 Prove Theorem 8.15.

8.9 A real number is said to be **algebraic** if it is a root of a polynomial equation

$$a_n x^n + \cdots + a_1 x + a_0 = 0$$

with integer coefficients. Note that the algebraic numbers include the rationals and all roots of rationals (such as $\sqrt{2}$, $\sqrt[3]{5}$, etc.). If a number is not algebraic, it is called **transcendental**.

- (a) Show that the set of polynomials with integer coefficients is countable.
 (b) Show that the set of algebraic numbers is countable.
 (c) Are there more algebraic numbers or transcendental numbers?

8.10 Prove: If S is denumerable, then S is equinumerous with a proper subset of itself.

8.11 Prove: Every infinite set has a denumerable subset. ☆

8.12 Prove: Every infinite set is equinumerous with a proper subset of itself.

8.13 Prove that our ordering of cardinal numbers is antisymmetric: if $|S| \leq |T|$ and $|T| \leq |S|$, then $|S| = |T|$. This result is known as the Schröder-Bernstein theorem and is very useful in proving sets equinumerous. (The proof is hard, but the hint in the back of the book will help.) ☆

8.14 Use the Schröder-Bernstein theorem (Exercise 8.13) to do parts (b) and (c) of Exercise 8.3.

8.15 Suppose that $A \subseteq B \subseteq C$ and $A \sim C$. Prove that $A \sim B$ and $B \sim C$.

- (e) $|S| \leq |T|$ means that there exists an injection $f: S \rightarrow T$.
- (f) The continuum hypothesis says that \aleph_0 is the smallest transfinite cardinal number.
- 8.3 Show that the following pairs of sets S and T are equinumerous by finding a specific bijection between the sets in each pair.
- (a) $S = [0, 1]$ and $T = [1, 3]$
- (b) $S = [0, 1]$ and $T = [0, 1)$ ☆
- (c) $S = [0, 1)$ and $T = (0, 1)$
- (d) $S = (0, 1)$ and $T = (0, \infty)$
- (e) $S = (0, 1)$ and $T = \mathbb{R}$
- 8.4 (a) Suppose that $m < n$. Prove that the intervals $(0, 1)$ and (m, n) are equinumerous by finding a specific bijection between them.
- (b) Use part (a) to prove that any two open intervals are equinumerous.
- 8.5 Prove: If $(S \setminus T) \sim (T \setminus S)$, then $S \sim T$. ☆
- 8.6 Prove: Every subset of a finite set is finite.
- 8.7 Use Example 8.11(d) to prove that \mathbb{Q} is countable.
- 8.8 Prove Theorem 8.15.
- 8.9 A real number is said to be **algebraic** if it is a root of a polynomial equation
- $$a_n x^n + \cdots + a_1 x + a_0 = 0$$
- with integer coefficients. Note that the algebraic numbers include the rationals and all roots of rationals (such as $\sqrt{2}$, $\sqrt[3]{5}$, etc.). If a number is not algebraic, it is called **transcendental**.
- (a) Show that the set of polynomials with integer coefficients is countable.
- (b) Show that the set of algebraic numbers is countable.
- (c) Are there more algebraic numbers or transcendental numbers?
- 8.10 Prove: If S is denumerable, then S is equinumerous with a proper subset of itself.
- 8.11 Prove: Every infinite set has a denumerable subset. ☆
- 8.12 Prove: Every infinite set is equinumerous with a proper subset of itself.
- 8.13 Prove that our ordering of cardinal numbers is antisymmetric: if $|S| \leq |T|$ and $|T| \leq |S|$, then $|S| = |T|$. This result is known as the Schröder-Bernstein theorem and is very useful in proving sets equinumerous. (The proof is hard, but the hint in the back of the book will help.) ☆
- 8.14 Use the Schröder-Bernstein theorem (Exercise 8.13) to do parts (b) and (c) of Exercise 8.3.
- 8.15 Suppose that $A \subseteq B \subseteq C$ and $A \sim C$. Prove that $A \sim B$ and $B \sim C$.

- 8.16 Determine whether each of the following is True or False. Explain your answers.
- (a) For every set S , $\emptyset \subseteq \mathcal{P}(S)$.
- (b) For every set S , $\emptyset \in \mathcal{P}(S)$.
- (c) $\{2\} \subseteq \mathcal{P}(\{2, 3\})$.
- (d) $\{2\} \in \mathcal{P}(\{2, 3\})$.
- (e) $\{\{2\}\} \subseteq \mathcal{P}(\{2, 3\})$.
- 8.17 Let A and B be sets. Prove $A \subseteq B$ iff $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.
- 8.18 Suppose that we let U denote the “set of all things.” Then for any set S we have $S \subseteq U$. In particular, $\mathcal{P}(U) \subseteq U$. Use Theorem 8.15 and Theorem 8.18 to obtain a contradiction. Thus we conclude that a giant universal set that contains “everything” is an impossibility.
- 8.19 (a) Prove: If $|S| \leq |T|$, then $|\mathcal{P}(S)| \leq |\mathcal{P}(T)|$.
- (b) Prove: If $|S| = |T|$, then $|\mathcal{P}(S)| = |\mathcal{P}(T)|$.
- 8.20 Is it possible for $\mathcal{P}(S) = \emptyset$ for some set S ? If yes, what can you say about S ? If no, explain why.
- 8.21 Let A and B be sets. Prove that $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.
- 8.22 Let A and B be sets. Prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$, and show by a counterexample that equality does not hold.
- 8.23 Let A and B be sets. Prove or give a counterexample: $\mathcal{P}(A \setminus B) = \mathcal{P}(A) \setminus \mathcal{P}(B)$.
- 8.24 In this exercise we outline a proof that $|\mathcal{P}(\mathbb{N})| = c$. By Exercise 8.13, it suffices to show that $|\mathcal{P}(\mathbb{N})| \leq c$ and $c \leq |\mathcal{P}(\mathbb{N})|$.
- (a) To show that $|\mathcal{P}(\mathbb{N})| \leq c$, we define a function $f: \mathcal{P}(\mathbb{N}) \rightarrow \mathbb{R}$ by $f(A) = 0.a_1 a_2 a_3 \cdots a_n \cdots$, where
- $$a_n = \begin{cases} 0, & \text{if } n \notin A \\ 1, & \text{if } n \in A. \end{cases}$$
- Show that f is injective.
- (b) To show that $c \leq |\mathcal{P}(\mathbb{N})|$, we use Exercise 8.19(b) to conclude that $|\mathcal{P}(\mathbb{N})| = |\mathcal{P}(\mathbb{Q})|$, since $|\mathbb{N}| = |\mathbb{Q}|$. Thus it suffices to find an injection $f: \mathbb{R} \rightarrow \mathcal{P}(\mathbb{Q})$. Define $f(x) = \{y \in \mathbb{Q} : y < x\}$. Use the fact (Theorem 12.12) that given $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$ to show that f is injective.
- 8.25 Let α and β be cardinal numbers. The **cardinal sum** of α and β , denoted $\alpha + \beta$, is the cardinal $|A \cup B|$, where A and B are disjoint sets such that $|A| = \alpha$ and $|B| = \beta$.
- (a) Prove that the sum is well-defined. That is, if $|A| = |C|$, $|B| = |D|$, $A \cap B = \emptyset$, and $C \cap D = \emptyset$, then $|A \cup B| = |C \cup D|$.

- (b) Prove that the sum is commutative and associative. That is, for any cardinals α , β , and γ , we have $\alpha + \beta = \beta + \alpha$ and $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
- (c) Show that $n + \aleph_0 = \aleph_0$ for any finite cardinal n .
- (d) Show that $\aleph_0 + \aleph_0 = \aleph_0$.
- (e) Show that $\aleph_0 + c = c$. ☆
- (f) Show that $c + c = c$.
- 8.26** Let α and β be cardinal numbers. The **cardinal product** $\alpha\beta$ is defined to be the cardinal $|A \times B|$, where $|A| = \alpha$ and $|B| = \beta$.
- (a) Prove that the product is well-defined. That is, if $|A| = |C|$ and $|B| = |D|$, then $|A \times B| = |C \times D|$.
- (b) Prove that the product is commutative and associative and that the distributive law holds. That is, for any cardinals α , β , and γ , we have $\alpha\beta = \beta\alpha$, $\alpha(\beta\gamma) = (\alpha\beta)\gamma$, and $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.
- (c) Show that $0\alpha = 0$ for any cardinal α .
- (d) Show that $n\aleph_0 = \aleph_0$ for any finite cardinal n with $n \neq 0$.
- (e) Show that $\aleph_0\aleph_0 = \aleph_0$.
- (f) Show that $cc = c$.

Section 9 AXIOMS FOR SET THEORY[†]

Throughout this chapter we have used the concept of a set informally without really saying what sets are or what properties they have. While a definition of “set” is essentially impossible, it is possible to discuss properties of sets and to indicate things that *cannot* be sets. In this section we present a list of axioms (the Zermelo-Fraenkel axioms) from which set theory can be derived in a formal way. It is not our intent actually to do this formal derivation, but rather to indicate a foundation upon which set theory may be built. We begin by considering two paradoxes.

Paradoxes

In Section 5 we saw the utility of having a “universal” set U that focused our attention on a particular mathematical system. Then any set under consideration was a subset of U . It is natural to ask: Why not let U denote the “set of all things”? Then certainly U will contain everything we might

[†] This section is optional and may be omitted without loss of continuity.