Assignment 6, Solutions
Due November 23

Page 231 Problem 4 (Read Problem 3 first), 13, 14, 15
Page 237 Problems 1, 2b), 2d)

Suggested Problems: page 237 5,6

Page 231, Problem 4 Find two numbers such that their sum and the sum of their squares are given numbers; say, their sum is 20 and the sum of the squares is 208

Let the two numbers be $10 + x$ and $10 - x$ so that their sum is 20 and we need only think about the second conditions $(10 + x)^2 + (10 - x)^2 = 208$. Expanding and cancelling gives $200 + 2x^2 = 208$ from which we see $x = \pm 2$ but $x = 2$ is adequate here: the two numbers are 8 and 12.

Page 231, Problem 13 Which of the following diophantine equations cannot be solved.

(a) $6x + 51y = 22$
(b) $33x + 14y = 115$
(c) $14x + 35y = 93$

(a) We can compute the prime decompoisition of the coefficients: $6 = 2 \cdot 3$ and $51 = 3 \cdot 17$. Therefore $\gcd(6,51)=3$. But $3 \nmid 22$ and so there is no pair of integer solutions to the equation $6x + 51y = 22$
(b) Here $33 = 3 \cdot 11$ and $14 = 2 \cdot 7$ so that $\gcd(33,14)=1$ and $1 \mid 115$ so there is an integer pair solution.
(c) Again $14 = 2 \cdot 7$ and $35 = 5 \cdot 7$ so that $\gcd(14,35)=7$ and $7 \nmid 93$ and so there no integer pair solution.

Page 231, Problem 14 Determine all solutions in the integers of the following diophantine equations.

(a) $56x + 72y = 40$
(b) $24x + 138y = 18$
(c) $221x + 35y = 11$

(a) Here $\gcd(56,72)=8$ and $8 \mid 40$ and so there are integer solutions. By experimentation $x_0 = 2$ and $y_0 = -1$ is a solution. The general solution is

$$x = x_0 + \frac{72}{8}t = 2 + 9t \quad y = y_0 - \frac{56}{8}t = -1 - 7t$$

that is whenever $t$ is an integer $(x, y)$ as defined here is a solution and any integer solution of $56x + 72y = 40$ must be of this form for some integer $t$.
(b) Here $\gcd(24,138)=6$ and $6 \mid 18$ and so there are integer solutions. One solution is $x_0 = -5$ and $y_0 = 1$ and so the general solution is

$$x = x_0 + \frac{138}{6}t = -5 + 23t \quad y = y_0 - \frac{24}{6}t = 1 - 4t$$

where $t$ is any integer.
(c) Here $\gcd(221,35)=1$ and $1 \mid 11$ and so there is an integer solution. One solutions is $x_0 = 1$ and $y_0 = -6$ and the general solutions is

$$x = x_0 + \frac{35}{1}t = 1 + 35t \quad y = y_0 - \frac{221}{1}t = -6 - 221t$$
Page 231, Problem 15 Determine all solutions in the positive integers of the following diophantine equations.

(a) \(18x + 5y = 48\)
(b) \(123x + 57y = 30\)
(c) \(123x + 360y = 99\)

(a) Here \(\gcd(18,5)=1\) and \(1|48\) and so there is an integer solution. One solution is \(x_0 = 1\) and \(y_0 = 6\) so that the general solution is

\[
x = x_0 + \frac{5}{1}t = 1 + 5t \quad y = y_0 - \frac{18}{1}t = 6 - 18t
\]

However we are asked for positive solutions and we only get \(x\) and \(y\) both positive when \(t = 0\) because \(t \geq 1\) implies \(y \leq -12\) and \(t \leq -1\) implies \(x \leq -4\): \(x_0 = 1\) and \(y_0 = 6\) is the only positive solution.

(b) Here \(\gcd(123,57)=3\) and \(3|30\) so that there is an integer solution. One such solution is \(x_0 = -3\) and \(y_0 = 7\) so that the general solution is

\[
x = x_0 + \frac{57}{3}t = -3 + 19t \quad y = y_0 - \frac{123}{3}t = 7 - 41t
\]

Which of these is positive? For \(x \geq 0\) we need \(t \geq 1\) and for \(y \geq 0\) we need \(t \leq 0\). None. There are no positive integer solutions to this problem.

(c) Here \(\gcd(123,360)=3\) and \(3|99\) so that there is an integer solution. One such solution is \(x_0 = 33\) and \(y_0 = -11\) so that the general solution is

\[
x = x_0 + \frac{360}{3}t = 33 + 120t \quad y = y_0 - \frac{123}{3}t = -11 - 41t
\]

Which of these is positive? For \(x \geq 0\) we need \(t \geq 0\) and for \(y \geq 0\) we need \(t \leq -1\). There are no positive integer solutions to this problem. In actual fact it is clear from the initial equation that if \(x \geq 0\) and \(y \geq 0\) then \(123x + 360y > 99\) except when both \(x = 0\) and \(y = 0\) and that is not a solution.

Page 237, Problem 1 Let the cubic equation \(ax^3 + bx^2 + cx + d = 0\) have integer coefficients, \(a\), \(b\), \(c\) and \(d\).

(a) Prove that if this equation has a rational root \(r/s\) where \(r\) and \(s\) are relatively prime integers (\(\gcd(r,s)=1\)) then \(r|d\) and \(s|a\).

(b) Show that if \(a = 1\), every rational root of the cubic equation must be an integer that divides \(d\).

(a) We follow the hint and substitute \(x = r/s\) where \(\gcd(r,s)=1\). (This is an hypothesis that \(x\) is rational and an observation that if so then it can be written “in lowest terms.”) Then we multiply through the cubic equation by \(s^3\) to clear fractions:

\[
ar^3 + br^2s + crs^2 + ds^3 = 0.
\]

From this it follows that \(r|ds^3\) (for solve for \(ds^3\) and factor out an \(r\)). But \(\gcd(r,s)=1\) and so, by Euclid’s Lemma (Page 177 of the text) \(r|d\).

Similar reasoning shows that \(s|ar^3\) and \(s\) and \(r\) have no common factors and so \(s|a\).

(b) In the special case that \(a = 1\) we know that any rational solution \(r/s\) with \(\gcd(r,s)=1\) must have \(s|1\) and so \(s = \pm 1\) and so \(r/s\) is an integer \(\pm r\). Therefore any rational solutions must be an integer.
Page 237, Problem 2b Any rational solution \(r/s\) of \(32x^3 - 6x - 1 = 0\) must, if it exists have \(r|1\) and \(s|32\). The possible roots are \(\pm 1, \pm 1/2, \pm 1/4, \pm 1/8, \pm 1/16\) and \(\pm 1/32\) and these can be checked by substituting into the equation. There are shortcuts. For example if the root is any of \(\pm 1/8, \pm 1/16\) and \(\pm 1/32\) then \(|32x^3 - 6x| \leq 32|x|^3 + 6|x| < 1\) and so none of those possibilities are actually roots. Also \(\pm 1\) can be eliminated quickly. We check \(1/2\) and find that it is indeed a root; \(-1/2\) is not; \(1/4\) is not but \(-1/4\) is. This allows us to factor the cubic polynomial by dividing \(2x - 1\) or \(4x + 1\) or both

\[
32x^3 - 6x - 1 = (2x - 1)(4x + 1)^2
\]

so that \(-1/4\) is a double root.

Page 237, Problem 2d Any rational solution \(r/s\) of \(x^3 - 7x^2 + 20x - 24 = 0\) must be an integer by the previous problem (Problem 1b page 237) and it must be a factor of 24. It is easy to see any root \(r\) cannot be negative because \(r < 0\) implies \(x^3 - 7x^2 + 20x - 24 < 0\). We need check \(r = 1, 2, 3, 4, 6, 8, 12, 24\) but \(r = 1\) and \(r = 2\) do not work and \(r = 3\) is indeed a root. Therefore \(x - 3\) must factor the polynomial. Dividing we have \(x^3 - 7x^2 + 20x - 24 = (x - 3)(x^2 - 4x + 8)\). We can continue to check for integer roots but the factoring says we will be finding a root of \(x^2 - 4x + 8\) and so we need only check factors of 8 in our original list: \(r = 4, r = 8\) and neither is a root of \(x^2 - 4x + 8\). Indeed the other roots are given by the quadratic formula

\[
\frac{4 \pm \sqrt{16 - 4(8)}}{2}
\]

and these are not real numbers. Therefore \(r = 3\) is the only real root and only rational root.