5.2 Diophantus: Diophantus lived in Alexandria in times of Roman domination ca 250 A.D. and he is known to us through his book *Arithmetica*. Diophantus and Pappus (ca 300) represent a shortlived revival of Greek mathematics in a society that did not value math as the Greeks had done 500-750 years earlier.

Notable in *Arithmetica* is the introduction of short forms that anticipate algebra: ζ is his unknown $\Delta^{\Upsilon} = \zeta^2$; $K^{\Upsilon} = \zeta^3$ $\Delta^{\Upsilon}\Delta = \zeta^4$; $\Delta K^{\Upsilon} = \zeta^5$ and $KK^{\Upsilon} = \zeta^6$. Diophantus also introduced a minus sign which was an upside down ψ.

Example: $\Delta^{\Upsilon}\Delta 1 K^{\Upsilon} 12 M 13 \psi \Delta^{\Upsilon} 27 \zeta 13 = \zeta^4 + 12\zeta^3 - 27\zeta^2 - 27\zeta + 13$. Of course our modern decimal numbers have been used where Diophantus would use the Greek numbers on page 14: for example $13 = \iota\gamma$.

Arithmetica consisted of 13 books of which we only have the first 6. The remaining 7 cannot even be traced to even Arab times. They consist of problems with solutions. Many of the problems may have multiple solutions but Diophantus just finds one solution and does not consider the question of other solutions or uniqueness.

Example: Book I, Problem 18: Find three numbers such that the sum of any pair exceeds the third by a given amount; say the given excesses are 35, 40 and 45.

Solution: Let s be the sum of the three numbers. Then, if (1) denotes the first number we know that s -(1) = 35 +(1) or s-2(1) =35. Similarly s-2(2) =40 and s -2(3) =45. Add the three equations. $3s - 2s = 120$. So the sum of the numbers is 120. The first number is 85/2, the second is 40 and the third is 75/2.

5.3 Diophantine Equations: Diophantus is memorialized in modern mathematics jargon by the *diophantine equations*. Any equation for which integer solutions are sought is a diophantine equation. Therefore $x^n + y^n = z^n$ is a diophantine equation not because of the form of the equation but because the interesting solutions are integers (Pythagorean triples if $n = 2$). Diophantus himself was interested in nonnegative rational solutions and so the terminology is somewhat misleading.

Interest in diophantine equations can be traced to India and Aryabhata (ca. 476-550) and Brahmagupta (ca. 625). Indians may well have known of Greek mathematics since the time of Alexander the Great and knowledge of Diophantus is possible since the Roman Empire would have spread his fame. We know Aryabhata by his book *Aryabhatiya*. He calculated π correct to five places. Brahmagupta computed the area of a (convex) quadrilateral inscribed in a circle as

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where a, b, c and d are the side lengths and s is the semi-perimeter ($s = (a+b+c+d)/2$). Heron's formula for the area of a triangle is the special case of Brahmagupta's when $d = 0$. (Heron of Alexandria (ca. 1070 AD).) Brahmagupta also derived.

Theorem: *The linear equation*

$$ax + by = c$$

with integer coefficients $a \neq 0$, $b \neq 0$ and c has integer solutions x and y if and only if $gcd(a,b)|c$. In the event that there is an integer solution (x_0, y_0) then all other integer solutions are given by

$$x = x_0 + \frac{b}{d}t \quad y = y_0 - \frac{a}{d}t$$

for some integer t, where d = gcd(a,b).

Proof: It is obvious from the equation $ax + by = c$ that any common divisor of a and b is a divisor of c.

Conversely suppose that $d = \gcd(a, b)$ and $d|c$ so that $c = rd$ for some integer r. (We want to show there is an integer solution pair (x, y), that is $ax + by = c$.) Recall (page 175 of the text) that there exist integers x' and y' so that $ax' + by' = d$. Multiply by r: $a(rx') + b(ry') = rd = c$. We get the required equation by setting $x = x'r$ and $y = y'r$.

Now suppose that (x_0, y_0) is an integer solution pair. Then (x, y) is a second solution pair if and only if $a(x - x_0) + b(y - y_0) = 0$ or

$$\frac{a}{d}(x - x_0) = -\frac{b}{d}(y - y_0)$$

We know that $\gcd(a/d, b/d) = 1$ (which says a/d and b/d have no common factors) and so $\frac{b}{d}$ must divide $x - x_0$ (for all the prime factors of b/d are not factors of a/d and so they must be factors of $x - x_0$): $x - x_0 = tb/d$ for some integer t. Substituting $x - x_0 = tb/d$ into the above equation we find $y = y_0 - ta/d$. Conversely if $x = x_0 + tb/d$ and $y = y_0 - ta/d$ for some integer t then $ax + by = c$ follows from $ax_0 + by_0 = c$. $\square$

Notice that rational solutions are not interesting in the present context. For every rational x there is a rational solution y and conversely.

The following example is of the type first found in the work of Sun-Tsu (ca 250) in China.

Example: (Compare to page 229) There are certain things whose number is unknown. When divided by 2, the remainder is 1; when divided by 3, the remainder is 2 and when divided by 5 the remainder is 3. What will be the number of things?

Solution: Let the number be N, Then

$$\begin{aligned} N &= 2x + 1 \\ N &= 3y + 2 \\ N &= 5z + 3 \end{aligned}$$

This is a set of three linear equatins in 4 unknowns. Generically there will be infinitely many solutions but we are only interested in integer solutions. The first equations is $-2x + N = 1$ which is solvable because gcd(-2,1)=1. One solution is $x = 0$ and $N = 1$. The general solution is $N = 1 + 2r$ and $x = r$ where r is an arbitrary integer. Substitute this choice of N into the second equation. $2r - 3y = 1$. One solution is $r = 2$ and $y = 1$. The general solution is $r = 2 + 3s$ and $y = 1 + 2s$ where s is an arbitrary integer. In terms of N we have $N = 1 + 2(2 + 3s) = 5 + 6s$. (Such N satisfies the first two equations for arbitrary integer s.) Finally the third equation becomes $6s - 5z = -2$ which has particular solution $s = 3$ and $z = 4$ and so the general solution is $s = 3 + 5t$ and $z = 4 + 6t$. In terms of N we have $N = 5 + 6(3 + 5t) = 23 + 30t$. The solutions of the problem are $N = \ldots - 7, 23, 53, 83, 113, \ldots$.

Linear diophantine equations were found in Indian mathematics particularly that of Bhaskara (1114-1185) who wrote *Siddhanta Siromani* a text written in four parts the first parts are mathematical (arithmetic and algebra) Read in the text of his solution of

$221y + 65 = 195x$ which has solutions because gcd(221,-195)=13 and 13|65. (Bhaskara was the first to recognize solutions to quadratic equations that were negative as valid solutions.)

Later Commentators: Greek mathematics was not as well regarded in the Roman world as it had been in the heyday of the Greek world but a few mathematicians did arise. The book gives an interesting account of the first prominent woman mathematician to our knowledge Hypatia (370-415). Also of note is Pappus (ca 300 A.D.) who wrote the *Mathematical Collection* which was a synopsis of math of the day but with improvements and contributions by Pappus himself such as a generalization of the Pythagorean theorem. There is also his famous result quoted in most calculus books.

Pappus's Theorem. *If a region in the plane of area A is rotated around an axis in the plane not through the region then the volume of the solid of rotation so obtained is $2\pi RA$ where R is the distance from the center of mass of the region to the axis of ratation.*

Example: The volume of a doughnut or torus of radius R and cross sectional radius b is $2\pi R(\pi b^2)$.

A further result of Pappus is a geometric one. If one starts two sets A, B, C and a, b, c of collinear points and one joins them in pairs (Ab, Ac, Ba, Bc, Ca, Cb) then the six joins intersect in three points and those three points are collinear.

Read about the persecution of Hypatia (370-415) who edited Claudius Ptolemy's *Almagest* and a commentary on Diophantus's *Arithmetica.* and the destruction of the Library in Alexandria and decline in regard for Greek scholarship and also the few souls such as Anicius Boethius (ca. 475-525) who preserved some of that scholarship. Boethius wrote *De Institutione Arithmeticae* which transmitted some knowledge of the principles of formal arithmetic to the Middle Ages although as a work of mathematics it is a version of Nicomachus's *Introductio Arithmeticae* (written ca. 100 AD).

5.5 Math of the Near and Far East: The great Islamic prophet Mohammed lived c. 570-636 B.C. His followers, Moslems spread out from the area of Mecca and Medina to conquer from northern India all the way across Northern Africa and into Spain. They wreaked some havoc but they also brought tolerance and respect for learning that was forgotten in Europe.

Mohammed ibn Mûsâ al-Khowârizmî (ca 780-850) introduced the Hindu system of numeration using 9 digits and a 0. These numerals are still referred to as Arabic because of his influence. (The symbols for the nine digits were still evolving and were not yet like ours.) Europe also learned "algebra" from his *Hisâb al-jabr w'al muqâbalah* which is where our word "algebra" came from. The word *algorithm* is a corruption of his name.

In his book *Hisâb al-jabr w'al muqâbalah*, al-Khowârizmî solves 40 problems some of which involve solving quadratic equations. Again the quadratic equation is written using only nonnegative coefficients and only nonnegative solutions are sought. The three forms of the quadratic equation are $x^2 + ax = b$, $x^2 = ax + b$ and $x^2 + b = ax$. (If there is a positive coefficient $c > 0$ in front of the x^2 then we multiply through by c and let $y = cx$.) It is easiest to solve the first geometrically. To a square of side length x add two rectangles $a/2$ by x to on adjacent sides. The figure is almost a square except that a square of side length $a/2$ is missing. So we add $a^2/4$ to both sides. Algebraicly we have

$x^2 + ax + a^2/4 = b + a^2/4$ so that $x + a/2 = \sqrt{b + a^2/4}$. Picture page 230.

Negative roots can be traced to the Hindu mathematician Bhaskara (1114-1185) mentioned above.

Thabît ibn Korra (ca 836-901) translated many Greek works to Arabic. including Apollonius's *Conic Sections*, Nicomachus's *Introductio Arithmeticae* and Archimedes *The Measurement of a Circle* and *On the Sphere and Cylinder* and Euclid's *Elements*. The first translations of the *Elements* to reach Europe were translations of Thabît's.

Thabît also wrote em Book on the Determination of Amicable Numbers. Two integers are amicable if the set of the proper divisors of one adds to the other and conversely. The book computes out the example 220 and 284 are amicable pairs; 1184 and 1210 are also. See the text for the check that 220 and 284 are amicable. (220 has proper divisors 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110 and they sum to 284; 284 has proper divisors 1, 2, 4, 71, 142 and they sum to 220.) Thabît stated the following the following

Thabît's Rule If $n \geq 2$ and if $p = 3 \cdot 2^n - 1$ and $q = 3 \cdot 2^{n-1} - 1$ and $r = 9 \cdot 2^{2n-1} - 1$ are all prime numbers then $M = 2^n pq$ and $N = 2^n r$ are amicable.

Example: Choose $n = 3$ and we find $r = 143$ is not prime. Choose $n = 4$ and we get $p = 47$, $q = 23$ and $r = 1151$ which are all prime. and $M = 16 \cdot 23 \cdot 47 = 17296$ and $N = 16 \cdot 1151 = 18416$. If we add the factors of M we get

$$1 + 2 + 4 + 8 + 16 + 23 + 46 + 92 + 184 + 368 + 47 + 94 + 188 + 376 + 752$$
$$+1081 + 2162 + 4324 + 8648 = N$$

and the factors of N add to

$$1 + 2 + 4 + 8 + 16 + 1151 + 2302 + 4604 + 9208 = M$$

Thabît also generalized the Pythagorean theorem in an interesting way on Page 233.

Umar al-Khayyâmi, known to us as Omar Khayyam (1048-1123) was an astronomer and poet as well as mathematician. His poetry comes to us through his *Rubaiyat*.

XII A Book of Verses underneath the Bough,
A Jug of Wine, a Loaf of Bread–and Thou
Beside me singing in the Wilderness–
Oh, Wilderness were Paradise enow!

He also created a calendar so accurate that it requires just one day correction every 5000 years. He wrote *Commentaries on the Difficulties in Premises of Euclid's Book* where he tried to replace the fifth postulate with something simpler and more intuitive.

In *Treatise on Demonstrations of Problems of al-Jabra and al-Muqabalah,* he solved cubic equations by geometric means. His solutions involved cubics with positive coefficients and positive solutions. For example to solve $x^3 + qx = r$, he rewrites this as $x^3 + b^2x = b^2c$. He constructs the parabola $x^2 = by$ and the circle $x(x - c) + y^2 = 0$ of center $(c/2, 0)$ and radius $c/2$. The x-coordinate of the intersection of the two curves is a solution of the cubic because, if we multiply by b^2 in the equation of the circle, we get $0 = b^2x(x-c) + (by)^2 = b^2x(x-c) + x^4$ because of the equation of the parabola. Picture page 245(?)

A similar problem is 7(a), page 263. Solve $x^3 + d = ax$ $(a > 0,\ d > 0)$ or what is more convenient but equivalent is $x^3 + b^2c = b^2x$. The solution we are told is the intersection of $x^2 = by$ (which is a ??) with $x^2 = cx + y^2$ (which is a ??). One can see from the picture that there may be as many as 4 solutions.

Multiply the second equation by b^2: $b^2x^2 = b^2cx + b^2y^2 = b^2cx + x^4$ and there is a solutions $x = 0$ and up to three other solutions of the cubic but only the positive ones were recognized by Khayyam.

He came up with such solutions to 14 different cases of cubics with positive coefficient and positive solutions. He was aware that there could be 2 solutions but not that there could be 3. He thought that there was no solution of the cubic (comparable to the quadratic formula) and admittedly the solution had to wait almost 400 years.

Arab mathematics suffered a decline in the 12th and 13th centuries which was the time of the crusades. A good part of the crusades as far as Europe is concerned is that some of the great works were transmitted to Europe via Latin translations.

Read about Liu Hui's commentary written in 263 A.D. on the *Nine Chapters on the Mathematical Art* which was a victim of the 213 B.C. Burning of the Books and survives now only through a few fragments that have been pieced together. By 868 China began "printing" books by carving pages into wooden blocks and Hui's was printed in 1084.

Liu Hui also wrote *Sea Island Mathematical Manual* in which one problem is the

There is a sea island that is to be measured. Two poles 5 paces high are erected 1000 paces apart. The summit of the island and the top of the nearer pole line up with a point J on the ground 123 paces behind that pole. If a line is from the summit of the island is drawn through the top of the further pole then it intersects ground level at A, 127 paces behind the further pole. Find the distance to the island and the elevation of the summit.

Observe that if we knew one of the quantities then the other could be derived by trigonometry.

Solution: Picture Page 259.

Let K be the point 123 paces behind the further pole FE. If GD is the nearer pole then the triangles CHG and FEK are similar, as are CGF and FKA. Therefore

$$\frac{CH}{FE} = \frac{HG}{EK} = \frac{CG}{FK} = \frac{GF}{AK}$$

Of course CH is the elevation of the summit minus 30 and HG is the distance from the top of the nearer pole straight across to the island. We know GF is 1000 and AK is 4 and FE =5 so that CH is $5000/4 = 1250$ and the elevation is 1255 paces. Similarly EK is 123 and so HG is $123(1000)/4 = 30750$ paces.

Read about Genghis Khan and his grandson Kublai Khan conquered China by 1279 and his empire was the largest yet seen. This was a good period for math but activity dropped off after 1300 in China and by the time it revived there was substantial interaction with westerners who came as missionaries.

Negative numbers were used as a matter of course in China by 1250 and probably before. Pascal's triangle was known. What is Pascal's triangle? the binomial theorem?

The *Nine Chapters* provides the first evidence of a using matrices to solve linear equations. See Page 257 of the text.

Example: (Question 16(b), page 265): There are 9 equal pieces of gold and 11 equal pieces of silver. The two lots weigh the same. If one piece is removed from each lot and put in the other then the lot containing mainly gold weighs 13 ounces less than the lot containing mainly silver. Find the weigh of each piece of gold and silver.

Let x be the weight of a piece of gold and y be the weight of a piece of silver. $9x = 11y$ and $9x - x + y = 11y - y + x - 13$ or $7x = 9y - 13$. The Chinese just wrote, what we call now, the coefficient (or "augmented") matrix. What we would write in rows they wrote in columns presumably reflecting how the Chinese write.

$$\begin{array}{rr} 9 & 7 \\ -11 & -9 \\ 0 & -13 \end{array}$$

and the solution is each piece of gold weighs $x = 143/4$ and each piece of silver weighs $y = 117/4$ ounces.

Problem 3, Page 237 A number r is said to be constructible if a line segment of length $|r|$ can be contructed given a unit length (the number 1) and compass and straight edge construction. We know that r is constructible if it can be calculated from

0 and 1 by a finite number of additions and subtractions, multiplications and divisions and extractions of square roots.

Example: $\frac{5}{8}+\frac{37}{\sqrt{11}}-\sqrt{2\sqrt{5}-11\sqrt{3}}$

It can further be shown that

Theorem: *Any solution of the cubic equation $ax^3+bx^2+cx+d=0$ where $a\neq 0$, b, c and d are integers has a constructible solution if and only if it has a rational solution.*

Show that is is not possible to double the cube with straight edge and compass construction. That is, given a cube a^3 construct a cube with side length b where $b^3=2a^3$ Divide by a^3 to get $x^3=2$ Does this have a rational solution? No because if $x=m/n$ is a solution and gcd(m,n)=1 then $2|m^3$ implies $2|m$ and $2|n$. Therefore $2^{1/3}$ is not constructible.