Chapter 4: Euclidean Geometry

Philip of Macedonia 382-336 B.C. and his son Alexander the Great 356-323 B.C. conquered Greece, Egypt and subjugated the city states like Athens. Their armies were primarily Greek and so the Greek culture was dispersed over an even broader area (even to Mesopotamia and India) but different cities were favored. One was Alexandria which rapidly grew to be a large Egyptian port city of commerce, trade and learning. There was the Museum (seat of the Muses) and the Library. Scholars were supported at the Museum at royal expense and the Library was reknowned for its extensive collections (that were burnt in 641). Please read about the Museum and Library of Alexandria.

Euclid, Archimedes, Eratosthenes, Appolonius, Pappus, Claudius Ptolemy, Diophantus all studied in Alexandria.

Euclid wrote the *Elements of Geometry* in 13 books. It is the only (but not the first) systematic treatment of geometry to reach us. It was a standard textbook from 300 B.C. to the 1800's with 1000 editions. Still little is known about Euclid. See the text (page 137). Also note the famous quote of Proclus 410-485 A.D. who was a commentator on Euclid. He said to Ptolemy I there is "no royal road to geometry" when the ruler asked for a quicker way to learn geometry.

Euclid's *Elements* became the model for mathematical structure. He started with "Common Notions" such as what a point and line are (see page 140 of the text) and then he added postulates

1. A straight line can be drawn from any point to any other point.
2. A finite straight line can be continued continuously in a line.
3. A circle may be described with any center and radius.
4. All right angles are equal to one another
5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the the two straight lines, if produced indefinitely meet on that side on which are the angles less than two right angles.

The picture for the fifth postulate is

The idea behind the postulates was that all geometry could be derived from these simple beginnings and Euclid went about doing just that. It should be noted that Euclid used things he didn't state. For example there is nothing that assures that a line drawn through a point inside a cirlce and through another point outside, will actually meet the circle. Later mathematicians filled some gaps until David Hilbert 1862-1943 tried to fill out the postulates and common notions so that the results of geometry did really follow by deduction. (He wrote *Grundlagen der Geometrie* in 1899) See page 142 of our text.

Postulate 5 was very controversial because it did not seem as fundamental as the others and so there were many attempts to derive it from the other 4. Lobachevsky 1793-1856 showed that you could replace the fifth postulate with: given a line and a point not on the line there exist two lines through the point which are parallel to the given line. He went on from there and built an entirely consistent "non Euclidean" geometry. He and John Bolyai 1802-1860 who similarly constructed geometry based on the negation of the fifth postulate were derided and denied support for work that David Hilbert would later describe as " the most suggestive and notable achievement of the last century."

The book proceeds to outline the derivation of some of the fundamentals of geometry such as the construction of an equilateral triangle with any given side length. Like the book we shall take the congruence of two triangles who have two equal sides and the included angle as an axiom and not try to derive it using superposition as Euclid did. He further shows isosceles triangles have equal angles, that is, if two sides of a triangle are equal then the opposite angles are also equal. Pappus (ca 300 A.D.) gave the brief proof on page 145.

Along the way Euclid shows (Proposition 11) that, given a line and a point not on the line there exists a perpendicular to the line through the point.

The next result that our author Burton (on his way to discuss the fifth postulate) considers is

Proposition 16 *If one of the sides of a triangle is produced, then the exterior angle is greater than either opposite interior angle*

Proof: Let ABC be the triangle and extend the side BC through C to D to create the exterior angle. If E is the midpoint of AC then extend BE through E to F so that BE = EF. Then AEB is congruent to CEF by side-angle-side congruence. So interior angle ∠BAE is equal to ∠ECF. Then Euclid goes on to argue that ∠ECF is smaller than ∠ACD because it is in the picture (from page 145 of the text).

This picture and proof use something reasonable but something that is not stated in the postulates is used. To show this we build a different physical model of geometry than is usual: geometry on a sphere like the earth. A point is a point on the sphere but a line through two points is actually a great circle route determined by passing a plane through the two points and through the center of the sphere as well. Where that plane intersects the sphere is a straight line. This definition works with the first 4 postulates.

Now on our sphere, draw a triangle ABC with one side AB extending along the equator more than 90 longitude degrees and make C the north pole. Then the extension of BC to D creates an angle that is smaller than 90 degrees and so is smaller than an included angle: the angles at A and B are both right angles. If one constructs F then it may go outside the $\angle$ACD. Picture

It follows therefore that we must assume something new to get Proposition 16. We assume it.

Two lines are defined as parallel if they never meet. On a sphere there are no parallel lines but our assumption of Proposition 16 has done away with the spherical model. Proposition 16 assures that, given a line ℓ and a point P not on ℓ, there is a line parallel to ℓ through P, for construct a perpendicular to ℓ through any point (other than P) and then construct a perpendicular ℓ' through P to this new line. Then ℓ and ℓ' do not meet because if they did they would form a triangle and the exterior angle would have to be larger than the interior right angle but it is clearly just a right angle itself (Proposition 13 treats supplementary angles).

Proposition 27: *If two lines are cut by a transversal so that a pair of alternate interior angles are equal then the lines are parallel.*

Picture (page 147). **Proof:** Suppose not. Suppose that the two lines are not parallel

and therefore they meet at some point and form a triangle ABC. An exterior angle to ABC is one of our two alternate interior angles and so it must be larger than the either of the interior angles of ABC but of course one of those interior angle is one of our pair and so it equals the exterior angle. This contradiction proves that the two lines are parallel.

Now the converse. For the first time we need the parallel postulate 5.

Proposition 29: *A transversal falling on two parallel lines makes alternate interior angles that are equal. The sum of two interior angles on the same side of the transversal is two right angles.*

Picture: Page 148

Proof: Suppose that the alternate angles were not equal, say ∠a ¿ ∠c. Since ∠b = 180°- ∠a as supplementary angles (Proposition 13) ∠a+ ∠b= 180° ¿ ∠c + ∠b. The Parallel Postulate 5 then says that the two lines must meet on that side of the transversal, contradicting the fact that the lines are parallel.

It follows from Proposition 29: **Proposition 30:** *If two distinct lines are each parallel to a third line then they are parallel to each other.* This follows (see page 149) because parallel is equivalent to the equality of alternate interior angles. One consequence is that there is exactly one line parallel to a given line through a given point not on the line. We have already seen that there is such a line (construct a perpendicular to a perpendicular). If there were two lines parallel to the initial line through the same point then they would have to be parallel but parallel lines can't meet.

This leads us finally to

Proposition 32: The sum of the interior angles of a triangle equals two right angles.

Picture page 149.

Proof: Through vertex B of ABC construct a line parallel to AC. This creates at B, angles equal to ∠A and ∠C that along with ∠B create a 180° angle.

The "mousetrap" proof of the Pythagorean theorem appears on page 150. Two other proofs appear on pages 152-153 Interestingly the converse is also established. Suppose that there is a triangle ABC so that $AB^2 + AC^2 = BC^2$.

Picture from page 151.

Construct a right triangle ACD with one leg AC and the other leg the same length as AB. Then the two triangles have the same side lengths by the Pythagorean theorem. They are therefore congruent and so the initial triangle must be a right triangle.

Greek Geometric Algebra: The Greeks abandoned numbers in favor of lengths probably because they were uncomfortable with the idea of irrational numbers. Not being able to represent certain lengths as the ratio of integers they preferred to leave everything as lengths and argue geometricly.

Picture from the bottom of page 153. For example to solve the equation $ax = bc$ for the unknown x, the idea was to construct a rectangle the same area as a $b \times c$ rectangle. Extend one side AB of the $b \times c$ rectangle ABCD by a units to E. Extend the diagonal ED until it meets the extension of BC at F. Then CF is length x: $a/c = b/x$ by similar triangles and the $a \times x$ rectangle can be constructed on the other side of EF.

Quadratic equations could also be solved by geomtric methods. (Did anyone ever think of solving cubics?) We will consider the equation

$$x^2 + ax = b^2$$

It is not important that b^2 is a perfect square as long as it is positive. (Any positive constructible number can be written as a perfect square by teh method of Problem 3, Section 3.5. The book also treats the case of $x^2 + b^2 = ax$ and the methods are comparable.

Geometrically we want to construct a rectangle with one side length a and indeterminate other side x such that if we attach a square x^2 to the end of that rectangle the total area is b^2. Picture on page 156.

Algebraically the Babylonians solved this by setting $y = a + x$ so that

$$xy = b^2 \qquad \text{and} \qquad y - x = a$$

and then setting $y = z + a/2$, and $x = z - a/2$ etc. The Greeks argued as follows. Construct b (see Problem 12 Section 4.2). Then let AB be a line segment of length a and construct a perpendicular BE of length b at endpoint B. Let P be the midpoint of AB and draw a circular arc centered at P and through E so that it meets the extension of AB at Q. Define $x =$ BQ. Picture on page 156.

Now x is the solution to our problem. This is a consequence of the Pythagorean theorem

$$\left(\frac{a}{2}\right)^2 + b^2 = \left(\frac{a}{2} + x\right)^2$$

Euclid gave a different geometric derivation: historically correct and mathematically obtuse?

The Golden Ratio: The "golden section" of a line segment of length a is a portion x of a so that a is to x as x is to the remainder $a - x$:

$$\frac{a}{x} = \frac{x}{a-x}$$

Equivalently $a^2 = ax + x^2$ which is the same type of quadratic that we just solved with $b = a$. The solution is

$$x = \frac{a}{2}(-1 \pm \sqrt{5})$$

Taking the positive solution: the golden segment of a is $(a/2)(\sqrt{5} - 1)$. If $a = 1$ then $x \approx 0.61803398875$. The reciprocal of the golden segment for $a = 1$ is the golden ratio:

$$\frac{2}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{2}(= x + 1) =\approx 1.61803398875$$

Example: Golden Rectangles Construct a golden rectangle, that is a rectangle with the ratio of length to width as the golden ratio.

Picture page 157

Solution: Construct a line segment AB of unit length and a perpendicular BE at one end so that BE is length one also. Draw an arc with center P the midpoint of AB and radius PE and this cuts the extension of AB at Q and BQ is x. Draw a semicircle with center B and radius x cuts the line AQ at C (and Q). The rectangle with vertices BQE is a golden rectangle as is the one with vertices BCE and so is the rectangle with base AQ and passing through E. Another golden rectangle has base AC and height x and finally base AB and height x.

Remark: The Parthenon in Athens was constructed to be a golden rectangle. (See also Problem 21 in Section 3.3.)

Constructing Regular Polygons: The Pentagon: Is it possible to inscribe in a circle a polygon with equal sides using compass and straight edge? The equilateral triangle and the square are fairly straightforward to construct and once you have those bisecting angles allows you to double the number of sides to that a hexagon and octagon etc. are constructible. A seven sided polygon was not constructed by the ancients but a pentagon and decagon were constructed. We construct a decagon and thereby the pentagon.

To construct a decagon or pentagon the goal is construct a 72° or 36° angle because that is the size of angle that one side of the pentagon or decagon will subtend. Start with a line segment AB of length a and construct a golden segment AC of length $x = a(\sqrt{5} - 1)/2$ along AB. Picture page 159. (don't label DC with x yet).

Draw a circular arc of radius a and center A through B and from B measure off a chord of length x meeting the arc at D. We will show that the triangles ABD and DCB are similar. It is clear that $\angle$D $=$ $\angle$B $(= \alpha)$. We use the definition of the golden segment

$$\frac{a}{x} = \frac{x}{a-x} \qquad \text{or} \qquad \frac{AD}{DB} = \frac{DB}{CB}$$

This says that the corresponding sides of ABD and DCB are in the same ratio and since the contained angles are equal, this shows the triangles are similar.

Now we chase angles. We now know that ACD is isosceles with sidelength x and so $\angle$A $=$ $\angle$CDA $= \beta$ and $\alpha =$ $\angle$D $=$ $\angle$BCD. Looking at triangle ABD we see that $2\alpha + \beta = 180°$ and from triangle ACD we see that $\alpha = 2\beta$ and so we have $5\beta = 180°$ and so $\beta = 36°$.

It now follows that the circle of center A and radius a has a decagon inscribed if we cut off chords like BD of length x successively around the circle. A regular pentagon is constructed by ignoring alternate verices of the decagon.

The pentagon usually with the diagonals drawn in became the insignia of the Pythagoreans.

Carl Friedrich Gauss (1777-1855) proved that a regular polygon with n sides could be constructed by straight edge and compass if $n = 17$ and this was the first advance (1796) since the ancient Greeks. Later he extended the result to show that if n is a prime of the form $2^{2^k} + 1$ then the polygon could be constructed ($n = 257$ and $n = 65,537$ for example.)