Chapter 3: The Beginnings of Greek Mathematics

Greeks were not confined to a neighborhood or the Aegean and the Peloponnesis. They ranged far and wide including present day Italy and Sicily and many lived in North Africa including Egypt. This is of course before the expansion of the Roman Empire (Punic Wars, 3rd Century B.C. and the Second Punic War was 218-201 B.C.)

The Greeks introduced abstraction and gave logical arguments. For example Greeks were interested in triangles and not simply triangular fields.

Thales of Miletus (ca. 625-547) (Miletus is in Asia Minor). Thales is sometimes known as the father of geometry probably because he was the first to use the deductive method. He proved among other things (see page 83 of the text) that a triangle inscribed inside a semi-circle (with one side the diameter) must be a right triangle. He also observed that the sides similar triangles are proportional. We do not have any mathematical works directly attributable to Thales or Pythagoras. Euclid's *Elements* (Euclid 323-285) and Nicomachus of Gerasa's *Introductio Arithmeticae* written ca. 100 B.C. are two math texts which we depend on but neither gives credit to the originators and discoverers of the results. *Introductio Arithmeticae* unlike *Elements* was a popular textbook for teaching geometry. Thales is credited with using similar triangles to discover the height of a the Great Pyramid of Gizeh (2600 B.C.) He similarly estimated distance to a ship offshore from a tower above the shore. See page 84 of the text.

Pythagoras of Samos (580-500 B.C.; or 569-475 B.C.) (Samos is an Island in the northern Aegean near Asia Minor) His father was given honorary citizenship for bringing corn (!) in a time of famine. He visited Thales of Miletus (who was 45 or 50 years older than Pythagoras) who influenced him towards mathematics but was probably too old to be his teacher. Thales student Anaximander may have taught Pythagoras.

Pythagoras visited Egypt and apparently learned ffrom the priests there their refusal to eat beans, their refusal to wear even cloths made from animal skins, and their striving for purity were all customs that Pythagoras would later adopt. In 525 BC Cambyses II, the king of Persia, invaded Egypt; Pythagoras was taken prisoner and taken to Babylon. He traveled to Crete as well. He formed a school in Samos, the 'semicircle' of Pythagoras, but was not very well received and settled in Crotona in the eastern heel of Italy and he established a school which which was partly religious/mystic, philosophical and political. Pythagoreans had no personal possessions and were vegetarians.

(1) that at its deepest level, reality is mathematical in nature,

(2) that philosophy can be used for spiritual purification,

(3) that the soul can rise to union with the divine,

(4) that certain symbols have a mystical significance, and

(5) that all brothers of the order should observe strict loyalty and secrecy.

Pupils studied the "quadrivium" arithmetica, (number theory), geometria, harmonia (music) and astrologia. (as opposed to the "trivium" of rhetoric, logic and grammar also studied in the Middle Ages.) It seems that Pythagoras and his school believed in number mysticism. For example they observed that increasing the length of a string on a musical instrument by a "nice" fraction (2 or 3/2 or 4/3) gave a harmonious sound. Incidentally the Ionic Greek numbers that we discussed in Chapter 1 were not introduced until 450 B.C. which is 50 years after Pythagoras and it is guessed that numbers could have been given by patterns of dots. Pythagoreans were the "first" to think of numbers 1,2,3, ... as existing separately from objects ("two" as opposed to "two dog")

The numbers 1, 4, 9, 16, 25, ..., n^2, are square numbers and the numbers 1, 3, 6, 10, 15, 21, ... t_n. If we take two copies of the triangular number t_n and rearrange the dots then we get a rectangle $2t_n = n(n+1)$ so that $t_n = n(n+1)/2$. We also see that

$$1 + 2 + 3 + 4 \ldots + n = t_n = \frac{n(n+1)}{2} \tag{1}$$

There are several similar identities that the Greeks knew which we shall derive geometrically. First

$$1 + 3 + 5 + 7 + 9 + 11 + \ldots 2n - 1 = n^2 \tag{2}$$

(The sum of the first n odd numbers is n^2.) This follows from the observations that $n^2 - (n-1)^2 = 2n - 1$ which is the nth odd number. Geometrically this is

Add this up from $n = 1$ to n gives a telescoping sum on the left and the result follows. Nichomachus (ca 100 B.C.) is credited with the following result

$$1 + 2^3 + 3^3 + 4^3 + 5^3 + \ldots + n^3 = t_n^2 = \frac{n^2(n+1)^2}{4} \tag{3}$$

This can be derived from the following identity

$$[k(k-1)+1] + [k(k-1)+3] + [k(k-1)+5] + \ldots [k(k-1)+2k-1] = k^3$$

First lets check that this starting point is valid. There are k terms and so the left side is

$$kk(k-1) + 1 + 3 + 5 + \ldots + 2k - 1 = k^3$$

by identity (2). Now apply this identity for $1 \le k \le n$

$$\begin{aligned}
1 &= 1^3 \\
3 + 5 &= 2^3 \\
7 + 9 + 11 &= 3^3 \\
13 + 15 + 17 + 19 &= 4^3 \\
&\vdots \\
[n(n-1)+1] + [n(n-1)+3] + \ldots + [n(n-1)+2n-1] &= n^3
\end{aligned}$$

Now add all theses equations. On the right hand side we get the sum of all the cubes $1 + 2^3 + 3^3 + \ldots + n^3$. On the left side we the sum of all odd numbers from 1 to $n^2 + n - 1 = 2m - 1$ where $m = n(n+1)/2 = t_n$ they add up to m^2.

$$t_n)^2 = 1 + 2^3 + 3^3 + 4^3 + 5^3 + \ldots + n^3$$

The ancient Greeks also had a formula for the sum of squares. Geometrically we place squares for the first square numbers, side by side. If we complete the array out to get a rectangular array then we add $1 + 3 + 6 + 10 + \ldots + t_{n-1}$ more dots:

$$\begin{aligned} 1 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots + n^2 + 1 + 3 + 6 + \ldots t_n &= (1 + 2 + 3 + \ldots + n)(n+1) \\ &= t_n(n+1) = \frac{n(n+1)^2}{2} \end{aligned}$$

or if S is the sum of squares

$$\begin{aligned} S + [\frac{1(2)}{2} + \frac{2(3)}{2} + \ldots \frac{n(n+1)}{2}] &= \frac{n(n+1)^2}{2} \\ S + \frac{1}{2}[1 + 1 + 2^2 + 2 + 3^2 + 3 + \ldots n^2 + n] &= \frac{n(n+1)^2}{2} \\ \frac{3}{2}S + \frac{n(n+1)}{4} &= \frac{n(n+1)^2}{2} \end{aligned}$$

Solving for S gives $S = 2/3n(n+1)(2n+1)/4 = \frac{1}{6}n(n+1)(2n+1)$. which is a well known formula.

Zeno and the Infinite Zeno had his own school in Elea in southwestern Italy ca. 450 B.C. He is famous for the following paradox. Suppose that the Achilles runs 10 yards/ sec and the tortoise runs 1 yard/sec and the tortoise has a 100 yard headstart. After 10 seconds Achilles reachs the starting place of the tortoise but the tortoise has moved on 10 yards. In another second Achilles reaches where the tortoise had reached but the tortoise has moved 1 yard further. And so on. Achilles keeps reaching where the tortoise had been but the tortoise has moved on and this goes on forever and so he never catches the tortoise. In our notation Achilles has run

$$10 + 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots = 11.111\ldots$$

minutes. Zeno felt that this was ludicrous: Achilles must run forever but in fact the sums add up to 11 + 1/9 minus that last term added. We say the sums converge.

3.3. The Pythagorean Theorem: Establishing the Pythagorean Theorem

$$c^2 = a^2 + b^2$$

Proof 1: Rearrange the $(a+b)^2$ first as two squares of side length a and b and two $a \times b$ rectangles and then break the rectangles up into 4 $a \times b$ right triangles arrayed around the outside with the long c side pointing in. Picture on top of Page 101.

Proof 2. This time $(a+b)^2$ is made up with $a \times b$ rectangles surrounding a $(a-b)^2$. Cutting the rectangles in two along their diagonal and throwing away the corner halves we have a c^2 which can be rearranged into two $a \times b$ rectangles plus $(a-b)^2$ which can be placed into an arrangement that is a^2 plus b^2. Picture at the bottom of page 101 and page 102.

There is a third proof on page 150.

Pythagorean Triples: Three integers (a, b, c) form a Pythagorean triple of $a^2+b^2 = c^2$. The triple is said to be primitive if a and b have no common factors (g.c.d.$(a,b) = 1$.)

Therefore (3,4,5) and (8,6,10) are Pythagorean triples but the latter is not primitive. Pythagoras is believed to have discovered how to generate lots of Pythagorean triples by using the identity

$$(2n^2+2n+1)^2 = 4n^4+4n^2(2n+1)+(2n+1)^2 = 4n^2(n^2+2n+1)+(2n+1)^2 = (2n^2+2n)^2+(2n+1)^2$$

Therefore is $a = 2n+1$, $b = 2n^2+2n$ and $c = 2n^2+2n+1$ the (a,b,c) is a Pythagorean triple. This is a method to generate infinitely many Pythagorean triples but they need not be primitive. Euclid did better than Pythagoras by exploiting the identity $(s-t)^2 + 4st = (s+t)^2$. So if we make s and t perfect squares $s = m^2$ and $t = n^2$ and say $m > n$ are integers.

$$4m^2n^2 + (m^2-n^2)^2 = (m^2+n^2)^2$$

so that $a = 2mn$ and $b = m^2 - n^2$ and $c = m^2 + n^2$ then (a, b, c) is a Pythagorean triple. (Pythagoras method is a special case of this one obtained by setting $m = n+1$.) Arab mathematicians showed that all Pythagorean triples were of the form $(2kmn, k(m^2 - n^2), k(m^2+n^2)$, $1 \leq n < m$, $1 \leq k$.

Irrational Numbers. There is a proof in Euclid's *Elements* that $\sqrt{2}$ is irrational. However the result was apparently known to Aristotle and legend says that the Pythagorean school was very upset by the thought that a ruler and compass construction could create a length, specifically the diagonal of a square that was not a rational multiple of the side length of the square. The argument is this: Given a square ABCD suppose that the diagonal AC is a rational multiple m/n times the side length AB: AC $= m/n$AB. Suppose further that the fraction m/n is in its lowest terms: m and n have no common factors. By the Pythagorean theorem $AB^2 + BC^2 = AC^2$ and BC = AB so that $2 = m^2/n^2$ or $m^2 = 2n^2$. Therefore 2 divides m^2. This implies 2 divides m. Therefore 2 divides n^2. This implies 2 divides n and this is a contradiction since m/n is in its lowest terms. So 2 is not rational. How do we check the statement: 2 divides m^2 then 2 divides m? It is in fact true that whenever a prime number p divides ab where a and b are integers then

p divides a or p divides b or both. (page 171 of the text). We can check our case a little more simply: If 2 does not divide m then $m = 2k + 1$ and $m^2 = 4k^2 + 4k + 1$ which is odd: so 2 divides m^2 implies 2 divides m. It is also true that the square root of any prime number is irrational.

Geometric Proof This argument is taken form Page 110 of the text. Let us suppose that the side length s_1 is commensurate with the diagonal d_1 of the square ABCD. Choose E on d_1 which is s_1 units from A. Define $s_2 = d_1 - s_1$ which is of course the remaining part of the diagonal and is commensurate with s_1 and d_1. At E construct a perpendicular to d_1 that crosses BC at F in a 45 degree angle. EC=EF $=s_2$. Also by congruent right triangles EF = BF (two side of the triangles are the same and so the third is also by the Pythagorean theorem). Construct a second square EFGC. The diagonal d_2 of that new square is $s_1 - s_2$ and is therefore commensurate. We want to show that this process of constructing squares goes on forever giving an infinite sequence of commensurate lengths that get smaller and smaller. In the second square cut off from the diagonal d_2 the length s_2 leaving behind $s_3 = d_2 - s_2$ which is commensurate. And so on. Picture Algebraicly we have $s_n = d_{n-1} - sn - 1$, $d_n = s_{n-1} - s_n = 2s_{n-1} - d_{n-1}$ and $\frac{s_1}{d_1} = \frac{M_1}{N_1}$

with $M_1 < N_1 < 2M_1$ by the Pythagorean theorem. Similarly $\frac{s_n}{d_n} = \frac{M_n}{N_n}$

$$\frac{M_n}{N_n} = \frac{s_n}{d_n} = \frac{d_{n-1} - sn - 1}{2s_{n-1} - d_{n-1}} = \frac{N_{n-1} - M_{n-1}}{2M_{n-1} - N_{n-1}}$$

but the numerator is $M_n = N_{n-1} - M_{n-1} < M_{n-1}$ and the denominator is $N_n = 2M_{n-1} - N_{n-1} < N_{n-1}$ The process continues and eventually $M_n = 0$ but that is impossible because the process continues forever.

Alternative argument: By the similarity of squares $M_2/N_2 = M_1/N_1$. This leads to $d_1^2 = 2s_1^2$ but now an algebraic argument is needed to give the contradiction.

Approximating $\sqrt{2}$: Theon (ca. 130 B.C.) is credited with the following iteration scheme for getting ever improving approximations of $\sqrt{2}$

$$\begin{array}{rclrcl} x_2 &=& x_1 + y_1, & y_2 &=& 2x_1 + y_1 \\ x_3 &=& x_2 + y_2, & y_3 &=& 2x_2 + y_2 \\ &\vdots& & &\vdots& \\ x_n &=& x_{n-1} + y_{n-1}, & y_n &=& 2x_{n-1} + y_{n-1} \end{array}$$

If we initiate the sequences with a pair of positive integers (x_1, y_1) that lie on either of the hyperbolas $y^2 - 2x^2 = \pm 1$, then y_n/x_n converges to $\sqrt{2}$. To show this we first observe that if (x_{n-1}, y_{n-1}) lie on one of the hyperbolas then (x_n, y_n) lies on the other because

$$y_n^2 - 2x_n^2 = (2x_{n-1} + y_{n-1})^2 - 2(x_{n-1} + y_{n-1})^2 = -(y_{n-1}^2 - 2x_{n-1}^2)$$

It is also clear that (x_n, y_n) gets "large" as n does and so our points are traveling out the hyperbolas. Picture

The book points out that the same scheme can approximate the square root of any integer a if we just replace the 2 in the above equations for the hyperbolas with a. (page 107). None of these schemes are numerically efficient although they are ingenious.

We should remember at this stage the linear approximation from calculus: If $f(x)$ is differentiable at x_0 the tangent line is $L(x) = f(x_0) + f'(x_0)(x - x_0)$. This is referred to as the linearization of $f(x)$ at x_0. If we take $f(x) = \sqrt{x}$ and $x_0 = a^2$ and $b = x - a^2$ we have $L(a^2 + b) = a + \frac{b}{2a}$ because $f'(x) = \frac{1}{2}x^{-1/2}$.

Example: Approximate $\sqrt{19}$. Write $19 = 4^2 + 3$ so that $\sqrt{19} \approx L(4^2 + 3) = 4 + 3/8$. We can refine this because $19 = (4 + 3/8)^2 - 9/64$ Therefore, letting $a = 4 + 3/8$ and $b = -9/64$

$$\sqrt{19} \approx 4 + \frac{3}{8} - \frac{9/64}{8 + 3/4} = 4 + \frac{201}{560} = 4.35892857$$

whereas a calculator gives $\sqrt{19} \approx 4.3588989$. This tangent line approximation for the square root function was used by Archimedes. See page 76 problems 7 and 8.

3.4 Three Construction Problems: Plato 429-348 B.C. is credited with the insistence that geometric constructions be done with a straight edge and compass only. That is a straight line may be drawn through any two points and a circle can be drawn with any given point for center and passing through any point. Neither the compass nor the straight edge can be used to transfer distances: the straight edge is not a ruler; the compass collapses when it is lifted off the page. Plato probably believed that anything could be constructed this way although possibly with difficulty and imposed this for aesthetic reasons.

The Three Construction Problems that came down to us from Greek times are

1. Quadrature of the circle. That is given a circle find a square with equal area.

2. Duplication of the Cube. That is given a cube construct another cube with twice the volume.

3. Trisection of the Angle. Given an arbitrary angle construct an angle one third as large.

All of the constructions of antiquity were by ruler and compass at first but if no construction was found then other methods were introduced. It was shown in the nineteenth century that none of these constructions are possible by ruler and compass alone. The proofs used algebra and it was shown that ruler and compass construction allows: given a unit length you can construct another length that is an arbitrary rational multiple of that length and one can also find the square root af any length. Therefore a length $(153/37)^{1/4}$ units is constructible. For the first problem one is trying to construct $\sqrt{\pi}$; for the second $2^{1/3}$.

Of course if one allows a broader variety of construction methods these problems are solvable. Hippocrates of Chios (460-380 B.C.) (who was not his contemporary Hippocrates of Cos, famous for the Hippocratic oath in medicine) was one of the foremeost geometers of his time contributed to the first two problems. For the first he constructed lunes of equal areas (a lune is a region bounded by the arcs of two circles.) For the second he showed that the problem reduced to constructing from a given number a, two quantities x and y so that

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{2a}$$

Because, if one constructs x and y then $x^2 = ay$ and $2ax = y^2$ so that $x^4 = (ay)^2 = 2a^3x$ so that $x = 2^{1/3}a$ is the sidelength of the cube with volume $2a^3$. This is the method that has been used repeatedly to double the cube although Hippocrates is not credited with having done so himself.

Pierre Wantzel (1814-1848) proved in 1837 in *Journal de Mathematiques* that one could not trisect the angle or double the cube by ruler and comapss construction. As for squaring the circle the futility of ruler and compass construction depends on the fact that $\sqrt{\pi}$ is not constructible and that depends on the fact π is transcendental; in other words it is not the root of any polynomial equation with integer coefficients. This latter fact was established by Ferdinand Lindemann (1852-1939) in 1882.

Read the description on page 121 of how Archimedes trisected an arbitray angle using a ruler (not straight edge) and compass.

The Quadratrix of Hippias: Please briefly read this sections and answer questions 2 and 3 on page 131. Question 3 does not depend on the readings.