This section is about the concept of distance in $\mathbb{R}$ and the related concepts of open and closed.

### 0.13 Topology of $\mathbb{R}$

**Definition:** If $x \in \mathbb{R}$ and $\epsilon > 0$ then the open interval $N(x;\epsilon) = (x - \epsilon, x + \epsilon) = \{y : |y - x| < \epsilon\}$ is an $\epsilon$-neighborhood of $x$. We further define $N^*(x;\epsilon) = N(x,\epsilon) \setminus \{x\} = \{y : 0 < |y - x| < \epsilon\}$ to be the deleted $\epsilon$-neighborhood of $x$.

**Picture:** These are the points within a distance of $\epsilon$ of $x$.

**Definition:** Let $S \subseteq \mathbb{R}$. If $x \in S$ then $x$ is said to be an interior point of $S$ if there exists and $\epsilon > 0$ so that $N(x;\epsilon) \subseteq S$. The set of all interior points of $S$ is the interior of $S$ and is denoted $\text{int}(S)$. If $x \in \mathbb{R}$ then $x$ is said to be a boundary point of $S$ if, for all $\epsilon > 0$ $N(x,\epsilon) \cap S \neq \emptyset$ and $N(x,\epsilon) \cap S^c \neq \emptyset$. The set of all boundary points of $S$ from the boundary of $S$ and is denoted $\text{bd}(S)$.

Observe that no point can be both an interior and boundary point of $S$.

**Examples:** Find $\text{int} S$ and $\text{bd} S$ if $S = (0, 1]$, or $S = (-2, 0) \cup (0, 1]$ or $S = \{1/n : n \in \mathbb{N}\}$.

**Definition:** A set $S \subseteq R$ is open if for every $x \in S$ there is $\epsilon > 0$ so that $N(x;\epsilon) \subseteq S$. A set $S$ is closed if its complement $S^c = \mathbb{R} \setminus S$ is open.

**Theorem 0.1.** A set $S$ is open iff $S = \text{int} S$. A set $S$ is closed iff $\text{bd}(S)$ is contained in $S$.

**Proof.** Observe in general that $\text{int} S \subseteq S$.

Now suppose that $S$ is open. Then for every $x \in S$, there is $\epsilon > 0$ so that $N(x,\epsilon) \subseteq S$ and this just says $x \in \text{int} S$ and since $x$ was arbitrary we have shown that $S \subseteq \text{int} S$ or equivalently $S = \text{int} S$.

Conversely suppose the $\text{int} S = S$. The if $x \in S$ then $x$ is an interior point and there is an $\epsilon > 0$ so that $N(x,\epsilon) \subseteq S$. But this says $S$ is open.

Now suppose that $S$ is closed and $x \in \text{bd}(S)$. Then for every $\epsilon > 0$ $N(x,\epsilon) \cap S^c \neq \emptyset$. But $S^c$ is open and so this implies that $x \notin S^c$ and that means $x \notin S$.

Conversely suppose $\text{bd}(S)$ is contained in $S$. Suppose that $x \in S^c$. Then there must exist $\epsilon > 0$ so that $N(x,\epsilon) \cap S = \emptyset$ for if not then $x$ belongs to $\text{bd}(S)$ and it can’t because $\text{bd}(S) \subseteq S$.

**Exercise:** Show that a set is open if and only if $S \cap \text{bd} S = \emptyset$.

**Example:** Is $S = (1, 3)$ open? Suppose that $x \in (1, 3)$ so that $1 < x < 3$. Choose $\epsilon = \min\{3 - x, x - 1\}$ the $N(x,\epsilon) \subseteq (1, 3)$. What is $\text{bd}(S)$? It is $\{1, 3\}$. This illustrates that an interval $(a, b)$ is open $[a, b]$ is closed; $(a, b]$ and $[a, b)$ are neither open nor closed.

The empty set is open. Why? $\mathbb{R}$ is also open. Why? It follows that $\emptyset$ is both open and closed (or clopen) and the same for $\mathbb{R}$. 

Theorem 0.2. The union $\cup_S S_\alpha$ of an arbitrary collection $\{S_\alpha : \alpha \in A\}$, where $A$ is just an index set, of open sets is open.

The intersection $\cap_{i=1}^n S_i$ of a finite number of open sets $S_i$, $1 \leq i \leq n$ is open.

Proof. Suppose $x \in \cup_{\alpha} S_\alpha$. Then there is $\alpha_0 \in A$ so that $x \in S_{\alpha_0}$. Because $S_{\alpha_0}$ is open there is an $\epsilon > 0$ so that $N(x, \epsilon) \subseteq S_{\alpha_0}$ but $S_{\alpha_0} \subseteq \cup_{\alpha} S_\alpha$ and so $N(x, \epsilon) \subseteq \cup_{\alpha} S_\alpha$ and this shows that $\cup_{\alpha} S_\alpha$ is open.

Suppose now that $x \in \cap_{i=1}^n S_i$. Then there exists $\epsilon_i > 0$ so that $N(x, \epsilon_i) \subseteq S_i$, for $1 \leq i \leq n$. Define $\epsilon = \min\{\epsilon_i : 1 \leq i \leq n\}$. Importantly $\epsilon > 0$. We also have $N(x, \epsilon) \subseteq N(x, \epsilon_i)$ for every $i$ and so $N(x, \epsilon) \subseteq \cap_{i=1}^n S_i$ and this verifies that $\cap_{i=1}^n S_i$ is open.

Example: Is $\mathbb{N} \subseteq \mathbb{R}$ open or closed or neither? Observe $\mathbb{N}^c$ is a union of unit intervals as well as $(-\infty, 1)$. Therefore $\mathbb{N}$, as a subset of $\mathbb{R}$, is closed.

Corollary 0.3. The intersection $\cap_{\alpha} S_\alpha$ of an arbitrary collection $\{S_\alpha : \alpha \in A\}$, where $A$ is just an index set, of closed sets is closed.

The union $\cap_{i=1}^n S_i$ of a finite number of closed sets $S_i$, $1 \leq i \leq n$ is closed.

Proof. Observe that $\cap_{\alpha} S_\alpha$ is closed if and only if its complement

$$(\cap_{\alpha} S_\alpha)^c = \cup_{\alpha \in A} S_\alpha^c$$

is open.

Example $\cap_{n \in \mathbb{N}}[−2^{-n}, 2^{-n}]$ is closed. What is it? $\cap_{n \in \mathbb{N}}(−2^{-n}, 2^{-n})$ need not be open. What is it? Give an example of an infinite collection of closed sets whose union is not closed.

Definition Let $S \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is said to be an accumulation point of $S$ if every deleted neighborhood $N^*(x, \epsilon)$ intersects $S$, that is for every $\epsilon > 0$, there exists $s \in S$ so that $0 < |s - x| < \epsilon$. The set of all accumulation points of $S$ is denoted $S'$.

Definition Let $S \subseteq \mathbb{R}$. Any point in $S \setminus S'$ is said to be isolated.

In other words $x \in S$ is isolated if there exists $\epsilon > 0$ so that $N(x, \epsilon) \cap S = \{x\}$.

Examples What if $S = [0, 1)$?

Suppose $S = (-1, 0) \cup \{1/n : n \in \mathbb{N}\}$. What is $S'$ and what is $S \setminus S'$? Sketch $S$. ($S' = [-1, 0]$.) What if $S = \{1/n : n \in \mathbb{N}\}$?

What if $S$ is a finite set: $S = \{x_1, x_2, x_3, \ldots, x_n\}$ then what is $S'$, int$S$, and what are the isolated points? ($S' = \emptyset = \text{int } S$ and every point of $S$ is isolated.

Definition The closure of a set $S \subseteq \mathbb{R}$ is the smallest closed set that contains $S$. In symbols $\text{cl } S$. Therefore $\text{cl } S \supseteq S$ and $\text{cl } S$ is closed and if $T$ is another closed set that contains $S$ then $\text{cl } S \subseteq T$.

Theorem 0.4. Let $S$ be a subset of $\mathbb{R}$. Then $\text{cl } S$ is a closed set and

$$\text{cl } S = S \cup \text{bd } S,$$

$$\text{cl } S = S \cup S'.$$
Proof. It is obvious from the definition that $\text{cl } S$ is a closed set.

Although $\text{bd } S$ and $S'$ are very different sets, they have the following common trait: If $U$ is an open set that intersects $\text{bd } S$ or $S'$ then $U$ intersects $S$. Let us check this property for $\text{bd } S$. Suppose that $x \in \text{bd } S \cap U$. Then there is $\epsilon > 0$ so that $N(x, \epsilon) \subseteq U$ because $U$ is open and $N(x, \epsilon)$ intersects $S$ because $x \in \text{bd } S$. Combining these facts we have $S$ intersects $U$.

A similar argument applies to $S'$. Suppose that $x \in S' \cap U$. Then there is $\epsilon > 0$ so that $N(x, \epsilon) \subseteq U$ because $U$ is open and $N(x, \epsilon)$ intersects $S$ because $x \in S'$. Therefore $U$ intersects $S$.

Let us now show that $S \cup \text{bd } S$ is closed. We do this by showing it contains its boundary. Suppose therefore that $x \in \text{bd } (S \cup S')$. Then, for all $\epsilon > 0$ $N(x, \epsilon)$ intersects $S \cup \text{bd } S$ but the above argument assures that the open set $N(x, \epsilon)$ cannot intersect $\text{bd } S$ without also intersecting $S$ and so $N(x, \epsilon)$ must intersect $S$ itself. So $x \in \text{bd } S$. This shows that $S \cup \text{bd } S$ is closed.

A similar argument applies to $S'$. Suppose therefore that $x \in \text{bd } (S \cup S')$. Then, for all $\epsilon > 0$ $N(x, \epsilon)$ intersects $S \cup S'$ but the above argument assures that the open set $N(x, \epsilon)$ cannot intersect $S'$ without also intersecting $S$ and so $N(x, \epsilon)$ must intersect $S$ itself. We see $S \cup S'$ contains its boundary and is therefore closed.

We can now prove the Theorem. We see immediately that $\text{cl } S \subseteq S \cup \text{bd } S$ because $\text{cl } S$ is the smallest closed set containing $S$. However if $\text{cl } S$ were a proper subset of $S \cup \text{bd } S$ then there would be $x \in S \cup \text{bd } S$ and $\epsilon > 0$ so that $N(x, \epsilon)$ did not intersect $\text{cl } S$. But this is impossible since either $x$ itself is in $S$ or $x \in \text{bd } S$ and so $N(x, \epsilon)$ must meet $S$ itself and therefore its closure.

The second part of the Theorem follows similarly. It is immediate that $\text{cl } S \subseteq S \cup S'$. However the containment cannot be proper because if $x \in S \cup S'$ but $x$ is not in $\text{cl } S$ then there is $\epsilon > 0$ so that $N(x, \epsilon)$ does not intersect $\text{cl } S$. On the other hand $x$ is either in $S$ or $S'$ and so either $x$ itself is in $S \subseteq \text{cl } S$ or $x \in S'$ in which case $N(x, \epsilon)$ must intersect $S$ and either is a contradiction. 

**Theorem 0.5.** Let $S$ be a subset of $\mathbb{R}$. The following are equivalent

1. $S$ is closed.
2. $S = \text{cl } S$
3. $S$ contains all its accumulation points.
4. $S$ contains all its boundary points.

**Proof.** $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$