This section is about the concept of distance in $\mathbb{R}$ and the related concepts of open and closed.

0.13 Topology of $\mathbb{R}$

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Definition: If $x \in \mathbb{R}$ and $\epsilon > 0$ then the open interval $N(x;\epsilon) = (x-\epsilon, x+\epsilon) = \{y : |y-x| < \epsilon\}$ is an *ϵ–neighborhood* of x. We further define $N^*(x;\epsilon) = N(x,\epsilon)\backslash\{x\} = \{y : 0 < |y-x| < \epsilon\}$ to be the deleted ϵ-neighborhood of x.

Picture: These are the points within a distance of ϵ of x.

Definition: Let $S \subseteq \mathbb{R}$. If $x \in S$ then x is said to be an interior point of S if there exists and $\epsilon > 0$ so that $N(x;\epsilon) \subseteq S$. The set of all interior points of S is the *interior* of S and is denoted int(S). If $x \in \mathbb{R}$ then x is said to be a boundary point of S if, for all $\epsilon > 0$ $N(x,\epsilon) \cap S \neq \emptyset$ and $N(x,\epsilon) \cap S^c \neq \emptyset$. The set of all boundary points of S froms the *boundary* of S and is denoted bd S.

Observe that no point can be both an interior and boundary point of S.

Examples: Find int S and bd S if $S = (0,1]$, or $S = (-2,0) \cup (0,1]$ or $S = \{1/n : n \in \mathbb{N}\}$

Definition:A set $S \subseteq R$ is open if for every $x \in S$ there is $\epsilon > 0$ so that $N(x;\epsilon) \subseteq S$. A set S is closed if its complement $S^c = \mathbb{R}\backslash S$ is open.

Theorem 0.1. *A set S is open iff S = int S. A set S is closed iff bd(S) is contained in S.*

Proof. Observe in general that int $S \subseteq S$.

Now suppose that S is open. Then for every $x \in S$, there is $\epsilon > 0$ so that $N(x,\epsilon) \subseteq S$ and this just says $x \in$ int$S$ and since x was arbitrary we have shown that $S \subseteq$ int S or equivalently $S =$ int S.

Conversely suppose the int $S = S$. The if $x \in S$ then x is an interior point and there is an $\epsilon > 0$ so that $N(x,\epsilon) \subseteq S$. But this says S is open.

Now suppose that S is closed and $x \in$ bd(S). Then for every $\epsilon > 0$ $N(x,\epsilon) \cap S^c \neq \emptyset$. But S^c is open and so this implies that $x \notin S^c$ and that means $x \in S$.

Conversely suppose bd(S) is contained in S. Suppose that $x \in S^c$. Then there must exist $\epsilon > 0$ so that $N(x,\epsilon) \cap S = \emptyset$ for if not then x belongs to bd(S) and it can't because bd$(S)$$\subseteq S$. ☐

Exercise: Show that a set is open if and only if $S \cap$ bd $S = \emptyset$

Example: Is $S = (1,3)$ open? Suppose that $x \in (1,3)$ so that $1 < x < 3$. Choose $\epsilon = min\{3-x, x-1\}$ the $N(x,\epsilon) \subseteq (1,3)$. What is bd$(S)$? It is $\{1,3\}$. This illustrates that an interval (a,b) is open $[a,b]$ is closed; $(a,b]$ and $[a,b)$ are neither open nor closed.

The empty set is open. Why? $\mathbb{R}$ is also open. Why? It follows that $\emptyset$ is both open and closed (or *clopen*) and the same for $\mathbb{R}$.

Theorem 0.2. *The union* $\cup_\alpha S_\alpha$ *of an arbitrary collection* $\{S_\alpha : \alpha \in A\}$, *where* A *is just an index set, of open sets is open.*

The intersection $\cap_{i=1}^n S_i$ *of a finite number of open sets* S_i, $1 \leq i \leq n$ *is open.*

Proof. Suppose $x \in \cup_\alpha S_\alpha$. Then there is $\alpha_0 \in A$ so that $x \in S_{\alpha_0}$. Because S_{α_0} is open there is an $\epsilon > 0$ so that $N(x, \epsilon) \subseteq S_{\alpha_0}$ but $S_{\alpha_0} \subseteq \cup_\alpha S_\alpha$ and so $N(x, \epsilon) \subseteq \cup_\alpha S_\alpha$ and this shows that $\cup_\alpha S_\alpha$ is open.

Suppose now that $x \in \cap_{i=1}^n S_i$. Then there exists $\epsilon_i > 0$ so that $N(\epsilon_i, x) \subseteq S_i$, for $1 \leq i \leq n$. Define $\epsilon = \min\{\epsilon_i : 1 \leq i \leq n\}$. Importantly $\epsilon > 0$. We also have $N(x, \epsilon) \subseteq N(x, \epsilon_i)$ for every i and so $N(x, \epsilon) \subseteq \cap_{i=1}^n S_i$ and this verifies that $\cap_{i=1}^n S_i$ is open. ☐

Example: Is $\mathbb{N} \subseteq \mathbb{R}$ open or closed or neither? Observe $\mathbb{N}^c$ is a union of unit intervals as well as $(-\infty, 1)$. Therefore $\mathbb{N}$, as a subset of $\mathbb{R}$, is closed.

Corollary 0.3. *The intersection* $\cap_\alpha S_\alpha$ *of an arbitrary collection* $\{S_\alpha : \alpha \in A\}$, *where* A *is just an index set, of closed sets is closed.*

The union $\cap_{i=1}^n S_i$ *of a finite number of closed sets* S_i, $1 \leq i \leq n$ *is closed.*

Proof. Observe that $\cap_\alpha S_\alpha$ is closed if and only if its complement

$$(\cap_\alpha S_\alpha)^c = \cup_{\alpha \in A} S_\alpha^c$$

is open. ☐

Example $\cap_{n\in\mathbb{N}}[-2^{-n}, 2^{-n}]$ is closed. What is it? $\cap_{n\in\mathbb{N}}(-2^{-n}, 2^{-n})$ need not be open. What is it? Give an example of an infinite collection of closed sets whose union is not closed.

Definition Let $S \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is said to be an *accumulation point* of S if every deleted neighborhood $N^*(x, \epsilon)$ intersects S, that is for every $\epsilon > 0$, there exists $s \in S$ so that $0 < |s - x| < \epsilon$. The set of all accumulation points of S is denotes S'.

Definition Let $S \subseteq \mathbb{R}$. Any point in $S \backslash S'$ is said to be isolated.

In other words $x \in S$ is isolated if there exists $\epsilon > 0$ so that $N(x, \epsilon) \cap S = \{x\}$.

Examples What if $S = [0, 1)$?

Suppose $S = (-1, 0) \cup \{\frac{1}{n} : n \in \mathbb{N}\}$. What is S' and what is $S \backslash S'$? Sketch S. ($S' = [-1, 0]$.) What if $S = \{\frac{1}{n} : n \in \mathbb{N}\}$?

What if S is a finite set: $S = \{x_1, x_2, x_3, \ldots, x_n\}$ then what is S', int$S$, and what are the isolated points? ($S' = \emptyset =$ int S and every point of S is isolated.

Definition The *closure* of a set $S \subseteq \mathbb{R}$ is the smallest closed set that contains S. In symbols cl S. Therefore cl $S \supseteq S$ and cl S is closed and if T is another closed set that contains S then cl $S \subseteq T$.

Theorem 0.4. *Let* S *be a subset of* $\mathbb{R}$. *Then cl* S *is a closed set and*

$$\begin{aligned} cl\, S &= S \cup bd\, S \\ cl\, S &= S \cup S' \end{aligned}$$

Proof. It is obvious from the definition that cl S is a closed set.

Although bd S and S' are very different sets, they have the following common trait: If U is an open set that intersects bd S or S' then U intersects S. Let us check this property for bd S. Suppose that $x \in \text{bd } S \cap U$. Then there is $\epsilon > 0$ so that $N(x, \epsilon) \subseteq U$ because U is open and $N(x, \epsilon)$ intersects S because $x \in \text{bd } S$. Combining these facts we have S intersects U.

A similar argument applies to S'. Suppose that $x \in S' \cap U$. Then there is $\epsilon > 0$ so that $N(x, \epsilon) \subseteq U$ because U is open and $N(x, \epsilon)$ intersects S because $x \in S'$. Therefore U intersect S.

Let us now show that $S \cup \text{bd } S$ is closed. We do this by showing it contains its boundary. Suppose therefore that $x \in \text{bd } (S \cup \text{bd } S)$. Then, for all $\epsilon > 0$ $N(x, \epsilon)$ intersects $S \cup \text{bd } S$ but the above argument assures that the open set $N(x, \epsilon)$ cannot intersect bd S without also intersecting S and so $N(x, \epsilon$ must intersect S itself. So $x \in \text{bd } S$. This shows that $S \cup \text{bd } S$ is closed.

A similar argument applies to S'.Suppose therefore that $x \in \text{bd } (S \cup S')$. Then, for all $\epsilon > 0$ $N(x, \epsilon)$ intersects $S \cup S'$ but the above argument assures that the open set $N(x, \epsilon)$ cannot intersect S' without also intersecting S and so $N(x, \epsilon)$ must intersect S itself. We see $S \cup S'$ contains its boundary and is therefore closed.

We can now prove the Theorem. We see immediately that $clS \subseteq S \cup \text{bd } S$ because cl S is the smallest closed set containing S. However if cl S were a proper subset of $S \cup \text{bd } S$ then there would be $x \in S \cup \text{bd } S$ and $\epsilon > 0$ so that $N(x, \epsilon)$ did not intersect cl S. But this is impossible since either x itself is in S or $x \in \text{bd } S$ and so $N(x, \epsilon)$ must meet S itself and therefore its closure.

The second part of the Theorem follows similarly. It is immediate that $clS \subseteq S \cup S'$. However the containment cannot be proper because if $x \in S \cup S'$ but x is not in cl S then there is $\epsilon > 0$ so that $N(x, \epsilon)$ does not intersect cl S. On the other hand x is either in S or S' and so either x itself is in $S \subseteq \text{cl } S$ or $x \in S'$ in which case $N(x, \epsilon)$ must intersect S and either is a contradiction. □

Theorem 0.5. *Let S be a subset of $\mathbb{R}$. The following are equivalent*

1. *S is closed.*
2. *$S = cl\ S$*
3. *S contains all its accumulation points.*
4. *S contains all its boundary points.*

Proof. $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$ □