Introduction to Mathematical Analysis

Math 3190

Text: Introduction to Analysis, 5th ed., by Edward D. Gaughan, American Mathematical Society, isbn-10: 0821847872 / isbn-13: 9780821847879 / Edition: 5th Revised edition

Reference: Analysis with an Introduction to Proof., 2nd edition, Steven R. Lay, Prentice Hall.

Catalogue Description: This course is intended to introduce students to higher mathematics. The techniques of proving theorems, including proofs by induction, will be emphasized. The course will include elementary set theory and equivalence relations and a discussion of the real number system. Proofs of some basic theorems from algebra, calculus or number theory will be studied.

Chapter -1: Logic and Proof: This initial material is drawn from Steven Lay's Chapter 1.

Section 1: Connectives the goal, at least initially, is to take complex, compound statements and decompose them into simpler parts with the goal of extablishing the truth or falsity of the initial compound statement in terms of the constituent parts. A sentence which can be classified as either true or false is said to be a **statement**. For example

The number 3 is even.

We know that even means being of the form $2n$ where n is an integer. Of course the example statement is false. Another example is

There are infinitely many pairs of integers $(p, p+2)$ so that both p and $p+2$ are prime.

Although no one knows whether this statement is true or false we all believe that it is one or the other. So it too is a statement. We do not include sentences like "This statement is false." We should also be aware that some statements like $x^2 + 2x \geq 0$ may be statements if the context is clear, that is if the domain of x is known. If the statement were " For all $x \geq 0$, $x^2 + 2x \geq 0$ then it is true but if no quantifier on x is given, then it may be false $x = -1$ for example. We will use p,q,r $\ldots$ to denote statements and we are interested in deciding the truth and falsity of compound statements made up of simpler statements by joining them with logical connectors. The logical connectors are as follows.

Negation: The negation of the statement "The number 3 is even" is "The number 3 is odd" because odd is equivalent to not even (not a multiple of 2). We write the negation of p and p with the understanding that p is true exactly when p is false and conversely $\tilde{p}$ is false exactly when p is true. We illustrate this with a simple **truth table**

p	$\sim p$
T	F
F	T

Statements like "She is my mother" can be negated by "She is not my mother" Others like "It is warm" may be more tricky and require a definition of warm. (Is "It is warm" a statement?)

Conjunction: The logical version of "and" is called conjunction.

Example: A cow is four legged and a ruminant. In other word to be a cow requires both having four legs and chewing the cud. Something that has four legs but does not chew its cud, like a dog is not a cow.

Another example is getting credit for this course requires both paying tuition and doing the course work.

Example:Suppose that f is continuous on the interval $[a, b]$ and differentiable on the interval (a, b). This is a conjunction of two statements that we encountered in Calculus I. If, for a particular f you can verify the conjunction then the Mean Values Theorem says: There exists c, $a < c < b$ so that $f(b) - f(a) = f'(c)(b - a)$ For example $f(x) = |x|$ on [-1,1] does not satisfy one of the hypotheses and does not satisfy the conclusion.

In symbols we write $p \wedge q$ for the conjunction of two statements p and q. The truth table in this case is

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Notice that the truth table contains $4 = 2^2$ rows because there are 2 choices for each of 2 statements

Disjunction: English has two forms of disjunction ("or"). I'll have Pepsi or Coke means one or the other and usually not both. ("exclusive disjunction.") In logic "or" includes both. I will ask my mother or father for the family car would mean that asking either the mother or the father is enough but asking both is possible as well. The statement $x \neq 0$ or $y \neq 0$ means $x^2 + y^2 > 0$.

In symbols we write $p \vee q$ for the disjoin of p and q. The truth table in this case is

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Implication: Example: If it rains then I will take an umbrella. Regrettably there are many ways to state an implication.

- f is differentiable implies f is continuous.
- If f is differentiable then f is continuous.
- f is continuous whenever it is differentiable. f is continuous if it is differentiable.
- f is differentiable only if it is continuous.
- For f to be continuous, it is sufficient for f to be differentiable.

- For f to be differentiable, it is necessary that f be continuous.

In symbols we write $p \Rightarrow q$. The truth table in this case is

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The understanding is that if the antecedent (the "if" part) is false then whether the consequent (the "then") part is true or false does not contradict the implication. For example. If you won a million dollars this morning then you will give each class mate one thousand dollars is regarded as a true statement simply because the hypothesis is almost certainly false. Of course if you had won the million and didn't share then it would be false.

Equivalence The compound statement $(p \Rightarrow q) \wedge (q \Rightarrow p)$ is abbreviated $p \Leftrightarrow q$ and is read p if and only if q or p is equivalent to q. For example f is an increasing function if and only if $-f$ is a decreasing function. The truth table in this case is

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Thus the statement $p \Leftrightarrow q$ is true exactly when p and q are either both true or both false.

Example: Show that $p \Rightarrow q \Leftrightarrow (\sim q \Rightarrow \sim p$

p	q	$p \Rightarrow q$	$\Leftrightarrow$	$\sim q$	$\Rightarrow$	$\sim p$
T	T	T	T	F	T	F
T	F	F	T	T	F	F
F	T	T	T	F	T	T
F	F	T	T	T	T	T

where we have shortened the table slightly by indicating the truth values under the equivalence sign.

Example: Show that if $n \in \mathbb{Z}$ (n is an integer) and n^2 is even then n is even.

Here p is the statement: A given integer squared n^2 is even; and q is the statement: n is even and we are to show $p \Rightarrow q$. We do a "proof by contradiction" or in other words $\sim q \Rightarrow \sim p$. By the preceding example these two are equivalent. Assume $\sim q$ that is n is not even or in other words n is odd. The means that $n = 2m + 1$ (an even integer plus one) for some integer m. Then $n^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$ and so is odd (an even integer plus one). Thus $(\sim q \Rightarrow \sim p)$ and so we have contradicted p namely that n^2 is even.

Exercise: Use proof by contradiction to show that, whenever n is an integer and 7 divides n^2 then 7 divides n.

Observe that 7 divides an integer m means $m = 7k$ for some integer k. For example 7 divides -42.

Section 2: Quantifiers: A sentence like $x - x^2 \leq 0$ is not a statement until x is specified. We therefore think of it as a statement $p(x)$. For example $p(2)$ is true but $p(1/2)$ is false.

There are two common quantifiers: Existential and Universal. For example the sentence, "There exists a real number x so that $x - x^2 \leq 0$ is a statement which is true. In symbols we write : $\exists x$ in $\mathbb{R}$ so that $x - x^2 \leq 0$. The sentence, "For all real x, $x - x^2 \leq 0$ " is also a statement but it is false. In symbols, we write: $\forall x$ in $\mathbb{R}$ $x - x^2 \leq 0$.

Examples: $\exists x$ so that $x^3 + x + 5 = 0$.

$\forall x$, $0 \leq x$, we have $x \geq \sin x$.

$\forall \epsilon > 0$, $\exists \delta > 0$ so that $0 < |x| < \delta$ implies $|\dfrac{\sin x}{x} - 1| < \epsilon$.

$\forall x > 0, \exists n$ in $\mathbb{N}$ (the whole numbers) so that $0 < 1/n < x < n$.

Negation It is often convenient to negate a compound statement.

Examples Negate the statement "Bob and Joe took their drivers test." to get "Bob or Joe did not take his drivers test." In general we have $\sim (p \wedge q)$ is equivalent to $(\sim p) \vee (\sim q)$. In other words conjunction changes to disjunction and conversely. This can be checked with a truth table.

p	q	$\sim (p \wedge q)$		$\Leftrightarrow$	$(\sim p) \vee (\sim q)$		
T	T	F	T	T	F	F	F
T	F	T	F	T	F	T	T
F	T	T	F	T	T	T	F
F	F	T	F	T	T	T	T

Similarly $\sim (p \vee q)$ is equivalent to $(\sim p) \wedge (\sim q)$.

Example: Negate the sentence: "Every American adult drives " "There is an American adult who does not drive" Observe that to negate "every" requires only once instance. To show that a statement about every American adult is false only requires finding one exception.

Example: $\forall x$ in $\mathbb{R}$, $x - x^2 \leq 0$ has negation $\exists x$ in $\mathbb{R}$ so that $x - x^2 > 0$. Notice the original statement is false and so the negation is true.

The negation of

$$\begin{array}{lcl} \forall x \text{ in } A, p(x) & \text{is} & \exists x \text{ in } A, \sim p(x) \\ \exists x \text{ in } A, p(x) & \text{is} & \forall x \text{ in } A, \sim p(x) \end{array}$$

Example: $\forall \epsilon > 0$, $\exists n > 0$ so that, for all $x > n$, $|\arctan x - \pi/2| < \epsilon$. The negation is: "$\exists \epsilon > 0$ so that for all $n > 0$ there is $x > n$ so that $|\arctan(x) - \pi/2| \geq \epsilon$.