Test 2, Math 1860 Section 011 or 012 (Circle One.)

Oct 2013 Review Name ______________________

A non graphing calculator and a formula sheet are permitted.

(20) 1. Do the *series* converge or diverge? Give reasons for your answer If a series converges, find its sum.

(a) $5 + \frac{10}{3} + \frac{20}{9} + \frac{40}{27} + \frac{80}{81} + \frac{160}{243} + \ldots$
This is a geometric series with $a = 5$ and $r = 2/3$. Because $-1 < r < 1$ the series converges to

$$5 + \frac{10}{3} + \frac{20}{9} + \frac{40}{27} + \frac{80}{81} + \frac{160}{243} + \ldots = \frac{a}{1-r} = \frac{5}{1-2/3} = 15$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 5(-1)^n}{3^n}$
This is the sum of two geometric series

$$\sum_{n=1}^{\infty} \frac{2^n + 5(-1)^n}{3^n} = \left[\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \ldots\right] + [-5/3 + 5/9 - 5/27 + \ldots]$$

The first series has $a = 2/3$ and $r = 2/3$ and so the first series converges ($-1 < r < 1$); the second series has $a = -5/3$ and $r = -1/3$ so that the second series also converges and

$$\sum_{n=1}^{\infty} \frac{2^n + 5(-1)^n}{3^n} = \frac{2/3}{1-2/3} + \frac{-5/3}{1-(-1/3)} = 3/4$$

(12) 2. Determine whether the integral is convergent or divergent. Evaluate it if it is convergent.

$$\int_1^{\infty} \frac{5}{(3+x)^{3/2}}\, dx$$

We have

$$\begin{aligned}\int_1^{\infty} \frac{5}{(3+x)^{3/2}}\, dx &= 5 \lim_{t\to\infty} \int_1^t (3+x)^{-3/2}\, dx \\ &= 5 \lim_{t\to\infty} \frac{1}{-1/2}(3+x)^{-1/2}|_1^t = -10 \lim_{t\to\infty}\left[\frac{1}{(3+t)^{1/2}} - \frac{1}{4^{1/2}}\right] = 5\end{aligned}$$

Therefore the integral is convergent to 5.

3. Determine whether the series is convergent or divergent. Explain your reasoning.

(24)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2+4}$

Solution: Compare the series to the p-series $\sum_{n=1}^{\infty} 3/n^2$ which is convergent because $p = 2 > 1$. Since $3/(n^2+4) < 3/n^2$ we see that the given series converges by the comparison test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^5+2n}}$

Compare to the p series $\sum_{n=1}^{\infty} 1/n^{1/2}$ where $p = 1/2 \leq 1$ and is therfore divergent. Try limit comparison

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{n^2}{\sqrt{n^5+2n}}}{1/n^{1/2}} &= \lim_{n\to\infty} \frac{n^2 n^{1/2}}{\sqrt{n^5+2n}} \\
&= \lim_{n\to\infty} \sqrt{\frac{n^5}{n^5+2n}} \\
&= \sqrt{\lim_{n\to\infty} \frac{n^5}{n^5}\frac{1}{1+2/n^5}} = \sqrt{\lim_{n\to\infty} \frac{1}{1+2/n^5}} = 1
\end{aligned}$$

Because the limit is 1 and $0 < 1 < \infty$ the two series converge or diverge together by the limit comparison test. Because $\sum_{n=1}^{\infty} 1/n^{1/2}$ diverges so does the given series.

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-1}{3n+5}\right)^n$

The root test applies here

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |\left(\frac{n-1}{3n+5}\right)^n|^{1/n} = \lim_{n\to\infty} \frac{n-1}{3n+5} = \frac{1}{3}$$

and since $1/3 < 1$ the series converges (absolutely).

Alternatively $(n-1)/(3n+5) \leq n/3n = 1/3$ and so by the comparison test applies

$$\sum_{n=1}^{\infty} \left(\frac{n-1}{3n+5}\right)^n \leq \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

and the latter series is sa geometric series with $r = 1/3$ which is convergent $(-1 < r < 1)$.

(d) $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n}$

The integral test works here but it is faster to use the ratio test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2 e^{-(n+1)}}{n^2 e^{-n}}\right| = \lim_{n\to\infty} \frac{(n+1)^2}{n^2}\frac{e^{-(n+1)}}{e^{-n}} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2}\frac{1}{e} = \frac{1}{e}$$

and $1/e < 1$ and so the series converges by the ratio test.

4. Determine whether the series is absolutely convergent, conditionally convergent or divergent. Explain your reasoning.

(18)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$

This is an alternating series and so the alternating series test applies. Since the $b_n = 1/(2n-1$ decrease to 0 as n goes to infinity, the alternating series test says that the series converges. Does it converge absolutely? That is, does the series $\sum_{n=1}^{\infty} 1/(2n-1)$ converge? Since $1/(2n-1) > 1/2n$ and the series $\sum_{n=1}^{\infty} 1/2n = (1/2)\sum_{n=1}^{\infty} 1/n$ diverges (harmonic series or p-series with $p=1$) so that $\sum_{n=1}^{\infty} 1/(2n-1)$ diverges by comparison and the original series converges conditionally but not absolutely.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n!}{10^n}$

Here the terms do not converge to 0: $\lim_{n\to\infty} n!/10^n = \infty$ (see section 10.1 of Thomas's 12th edition.) Therefore the series diverges. Alternately one could use the ratio test: $|a_{n+1}/a_n| = (n+1)/10$ which diverges.

5. **(a)** Find the series' radius and interval of convergence. For what values of x does the series converge **(b)** absolutely, **(c)** conditionally

(12)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+3)^n}{n2^n}$$

Apply the root test

$$\begin{aligned}
\lim_{n\to\infty} |a_n|^{1/n} &= \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(x+3)^n}{n2^n}\right|^{1/n} \\
&= \lim_{n\to\infty} \left|\frac{(x+3)^n}{n2^n}\right|^{1/n} \\
&= \lim_{n\to\infty} \frac{|x+3|}{n^{1/n}2} = \frac{|x+3|}{2} \lim_{n\to\infty} \frac{1}{n^{1/n}} = \frac{|x+3|}{2}
\end{aligned}$$

By the root test the series converges absolutely if $|x+3|/2 < 1$ which means $-2 < x+3 < 2$ or $-5 < x < -1$ and it diverges if $x < -5$ or $x > -1$. The radius of convergence is 2 (half the length of the interval). Check the end points. If $x = -1$ then the power series is

$$\sum_{n=2}^{\infty} \frac{(-1)^{n+1}(-1+3)^n}{n2^n} = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}2^n}{n2^n} = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n}$$

which is the alternating harmonic series and it converges by the alternating series test or by an example done in class. If $x = -5$ then the power series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-5+3)^n}{n2^n} \sum_{n=2}^{\infty} \frac{(-1)^{2n+1}2^n}{n2^n} = \sum_{n=2}^{\infty} \frac{-1}{n}$$

and this is (-1) time the harmonic series and so it is divergent. The interval of convergence for the power series is therefore $-5 < x \leq -1$.