 Test 2B, Math 1860-022 100 total points
3/27/14 Solutions Name ________________
The percentage value of each question is indicated in the margin.

1. Express the integrand as a sum of partial fractions and evaluate the integral

$$\int \frac{2t^2+2t-1}{t^3+t^2}\,dt$$

(17)

Apply partial fractions. The degree on the top is 2 which is strictly smaller than the degree on bottom and so division is not required. We factor the bottom and expand by partial fractions

$$\frac{2t^2+2t-1}{t^3+t^2} = \frac{2t^2+2t-1}{t^2(t+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1}$$

Solve for the constants: Multiply both sides $t^2(t+1)$: $2t^2+2t-1 = At(t+1) + B(t+1) + Ct^2 = (A+C)t^2 + (A+B)t + B$. Comparing the coefficients we have

$$\begin{array}{ccccccc} A & & & + & C & = & 2 \\ A & + & B & & & = & 2 \\ & & B & & & = & -1 \end{array}$$

so that $B = -1$, $A = 3$ and $C = -1$. Therefore

$$\frac{4t^2+t-2}{t^2(t+1)} = \frac{3}{t} - \frac{1}{t^2} - \frac{1}{t+1}$$

This can be checked by finding a common denominator:

$$\frac{3}{t} - \frac{1}{t^2} - \frac{1}{t+1} = \frac{3t(t+1)-(t+1)-t^2}{t^2(t+1)} = \frac{2t^2+2t-1}{t^2(t+1)}$$

It checks. Integrate.

$$\begin{aligned} \int \frac{2t^2+2t-1}{t^3+t^2}\,dt &= \int \frac{3}{t}\,dt - \int t^{-2}\,dt - \int \frac{1}{t+1}\,dt \\ &= 3\ln|t| + \frac{1}{t} - \ln|t+1| + C \end{aligned}$$

Check by differentiation:

$$\frac{d}{dt}[3\ln|t| + \frac{1}{t} - \ln|t+1|] = \frac{3}{t} - t^{-2} - \frac{1}{t+1}$$

and so it checks.

2. Evaluate the improper integral (if it exists) $\int_2^\infty \frac{1}{x^2+4}\,dx$ (12)

$$\begin{aligned}\int_2^\infty \frac{1}{x^2+4}\,dx &= \lim_{t\to\infty}\int_2^t \frac{1}{x^2+4}\,dx\\ &= \lim_{t\to\infty}\frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big|_2^t\\ &= \lim_{t\to\infty}\frac{1}{2}\left[\arctan\left(\frac{t}{2}\right)-\arctan(1)\right]\\ &= \frac{1}{2}\left[\frac{\pi}{2}-\frac{\pi}{4}\right]=\frac{\pi}{8}\end{aligned}$$

The improper integral exists and is $\pi/8$.

(12) 3. Determine whether the *series* converges or diverges. If it converges then find its sum.

$$\sum_{n=0}^\infty\left(\frac{5}{3^n}+\frac{(-1)^{n+1}}{4^n}\right)$$

This is the sum of two geometric series, one with $r=1/3$ and the other with $r=-1/4$ and so both are convergent (since $|r|<1$ in each case). Using the formula $a/(1-r)$ for the sum of a geometric series we have

$$\sum_{n=0}^\infty\left(\frac{5}{3^n}+\frac{(-1)^{n+1}}{4^n}\right)=\frac{5}{1-1/3}+\frac{-1}{1-(-1/4)}\frac{15}{2}-\frac{4}{5}=\frac{67}{10}$$

(9 ea) 4. Do the series converge or diverge? Give reasons for your answers.

(a) $\displaystyle\sum_{n=3}^\infty\sqrt{\frac{n-3}{n^5+2n^2+1}}$

Compare to the series $\sum_{n=3}^\infty\sqrt{\frac{n}{n^5}}=\sum_{n=3}^\infty\frac{1}{n^2}$ which is a p-series with $p=2>1$ and so is convergent. Try direct comparison.

$$\sqrt{\frac{n-3}{n^5+2n^2+1}}\le\sqrt{\frac{n}{n^5+2n+3}}\le\sqrt{\frac{n}{n^5}}=\frac{1}{n^2}$$

This shows that the given series is smaller than a convergent p-series and is therefore convergent by the Direct Comparison test. (Limit Comparison also applies.)

(b) $\displaystyle\sum_{n=2}^\infty\frac{(\sin n)^2}{n^2+5\sqrt{n}}$

We note that $(\sin n)^2\le 1$ and so comparison works here. We compare to $\sum_{n=2}^\infty\frac{1}{n^2}$ which is a p-series with $p=2>1$ and so is convergent. Try Direct Comparison

$$\frac{(\sin n)^2}{n^2+5n}\le\frac{1}{n^2+5n}\le\frac{1}{n^2}$$

and so the given series is smaller than a convergent p-series and so it must converge.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{n-3}{n^2+4n}$

Compare to the series $\sum_{n=2}^{\infty} \frac{1}{n}$ which is the harmonic series and it diverges. (It is also a p-series with $p = 1 \leq 1$ and so divergent.) Direct Comparision does not seem very simple here and so we try Limit Comparison.

$$\lim_{n\to\infty} \frac{\frac{n-3}{n^2+4n}}{1/n} = \lim_{n\to\infty} \frac{n(n-3)}{n^2+4n} = 1$$

(by l'Hospital's rule or by factoring n^2 from top and bottom). Since $0 < 1 < \infty$ Limit Comparision says that the series either both converge or both diverge. Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, the given series, $\displaystyle\sum_{n=2}^{\infty} \frac{n-3}{n^2+4n}$ also diverges.

5. Which of the series converge absolutely, converge (conditionally) or diverge? Give reasons for your answers. (16)

(a) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n \ln n}{n}$

It does not look like this series converges absolutely because

$$\sum_{n=3}^{\infty} \left| \frac{(-1)^n \ln n}{n} \right| = \sum_{n=3}^{\infty} \frac{\ln n}{n} \geq \sum_{n=3}^{\infty} \frac{1}{n}$$

The latter series is a p-series with $p = 1 \leq 1$ and so is divergent. It follows that the given series does not converge absolutely by Direct Comparison. To determine if it converges conditionally we apply the alternating series test. Check that the terms go to 0: by l'Hospital's Rule

$$\lim_{n\to\infty} \frac{\ln n}{n} \left(= \frac{\infty}{\infty}\right) = \lim_{n\to\infty} \frac{1/n}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$$

and also that it decreases:

$$\frac{d}{dn} \frac{\ln n}{n} = \frac{(1/n)n - \ln n}{n^2} = \frac{1 - \ln n}{n^2}$$

which is negative for $n > e$ and so the sequence of terms decreases once $n > e$. The alternating series test says the series converges at least conditionally and since we saw that it did not converge absolutely it is indeed only conditionally convergent.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n^{1/n}}$

We recall that $\lim_{n\to\infty} n^{1/n} = 1$. (See Section 10.1.) Therefore the terms in this series do not go to 0 and so it diverges by the nth term test for divergence. The alternating series test leads to the same conclusion.

(16) 6. (a) Find the series' radius and interval of convergence.

$$\sum_{n=1}^{\infty} \frac{1}{n2^n}(x-3)^n$$

The root test applies here.

$$\lim_{n\to\infty} \left| \frac{1}{n2^n}(x-3)^n \right|^{1/n} = \lim_{n\to\infty} \frac{1}{(n2^n)^{1/n}} |x-3|^n)^{1/n} = \lim_{n\to\infty} \frac{1}{n^{1/n}2}|x-3| = \frac{1}{2}|x-3|$$

and so the series converges absolutely, by the root test if $(1/2)|x-3| < 1$ and that is the same as $-2 < x-3 < 2$ or $1 < x < 5$. The radius of convergence is half the interval length and that is 2. It remains to check the end points, $x = 1$ and $x = 5$. If $x = 5$

$$\sum_{n=1}^{\infty} \frac{1}{n2^n}(x-3)^n = \sum_{n=1}^{\infty} \frac{1}{n2^n}(5-3)^n = \sum_{n=1}^{\infty} \frac{1}{n}$$

and that is the harmonic series which is divergent. If $x = 1$ then

$$\sum_{n=1}^{\infty} \frac{1}{n2^n}(x-3)^n = \sum_{n=1}^{\infty} \frac{1}{n2^n}(1-3)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

and that is the alternating harmonic series which converges (by the alternating series test) but only conditionally because the harmonic series diverges.

(b) For what values of x does the series above converge absolutely.

The series converges absolutely if $1 < x < 5$.

(c) For what values of x does the series above converge conditionally but not absolutely.

The series converges conditionally but not absolutely at $x = 1$.