 Test 1, Math 1860 Section 020
Feb. 2014 Solutions Name ____________________

1. Evaluate the integral and simplify your answer.

(6)

(a) $\int_{\ln 2}^{\ln 25} e^{-x/2}\,dx$

An antiderivative of $e^{-x/2}$ is $-2e^{-x/2}$ and so

$$\begin{aligned}\int_{\ln 2}^{\ln 25} e^{-x/2}\,dx &= -2e^{-x/2}|_{\ln 2}^{\ln 25} \\ &= -2e^{-(\ln 25)/2} + 2e^{-(\ln 2)/2} = -2e^{\ln(1/5)} + 2e^{\ln(1/\sqrt{2})} = \sqrt{2} - \frac{2}{5}\end{aligned}$$

(8)

(b) $\int_{-\pi}^{\pi} \frac{\sin x\,dx}{2+\cos x}$

Substitute $u = 2+\cos x$ so that $du = -\sin x$ and

$$\int \frac{\sin x\,dx}{2+\cos x} = -\int \frac{1}{u}\,du = -\ln|u| + C = -\ln|2+\cos x| + C$$

and this can be checked by differentiation: $-(d/dx)\ln|2+\cos x| = \sin x/(2+\cos x)$. Therefore

$$\int_0^{\pi} \frac{\sin x\,dx}{2+\cos x} = -\ln|2+\cos x||_0^{\pi} = -\ln|2+\cos(\pi)| + \ln|2+\cos 0| = -\ln 1 + \ln 3 = \ln 3$$

2. Evaluate the integral.

(a) $\int_1^e t^2 \ln t\,dt$

We integrate by parts (that is $\int u\,dv = uv - \int v\,du$). Let $u = \ln t$ and $dv = t^2\,dt$ so that $du = (1/t)\,dt$ and $v = t^3/3$. Therefore

$$\begin{aligned}\int_1^e t^2 \ln t\,dt &= \frac{1}{3}t^3 \ln t|_1^e - \int_1^e \frac{1}{3}t^3\frac{1}{t}\,dt \\ &= \frac{1}{3}e^3 \ln e - \frac{1}{3}\ln 1 - \frac{1}{3}\int_1^e t^2\,dt = \frac{e^3}{3} - \frac{1}{9}t^3|_1^e = \frac{2e^3+1}{9}\end{aligned}$$

(b) $\int x\cos 3x\,dx$

We again integrate by parts (that is $\int u\,dv = uv - \int v\,du$. Let $u = x$ and $dv = \cos 3x\,dx$ so that $du = dx$ and $v = (1/3)\sin 3x$. Therfore

$$\int x\cos 3x\,dx = \frac{1}{3}x\sin 3x - \int \frac{1}{3}\sin 3x\,dx = \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C$$

This can be checked by differentiation: $(d/dx)[\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x] = \frac{1}{3}\sin 3x + x\cos 3x - \frac{1}{3}\sin 3x$.

(c) $\int (\sin 3\theta)^2 \, d\theta$

Use a trig identity $(\sin 3\theta)^2 = (1/2)(1 - \cos 6\theta)$

$$\int (\sin 3\theta)^2 \, d\theta = \frac{1}{2} \int 1 - \cos 6\theta \, d\theta = \frac{1}{2}[\theta - \frac{1}{6} \sin 6\theta] + C = \frac{\theta}{2} - \frac{1}{12} \sin 6\theta + C$$

and this can be checked by differentiation $(d/d\theta)[\frac{\theta}{2} - (1/12) \sin 6\theta] = (1/2) - (1/2) \cos 6\theta = (\sin 3\theta)^2$ again using that trig identity.

(d) $\int \frac{x^3}{\sqrt{4 + x^2}} \, dx$

Trig substitution. Let $x = 2 \tan \theta$ so that $dx = 2(\sec \theta)^2 \, d\theta$ and $\sqrt{4 + x^2} = 2 \sec \theta$. Therefore

$$\int \frac{x^3}{\sqrt{4 + x^2}} \, dx = \int \frac{(2 \tan \theta)^3}{2 \sec \theta} 2(\sec \theta)^2 \, d\theta = 8 \int (\tan \theta)^3 \sec \theta \, d\theta$$

Let $u = \sec \theta$ so that $du = \sec \theta \tan \theta$ and use the identity $(\tan \theta)^2 = (\sec \theta)^2 - 1 = u^2 - 1$.

$$\begin{aligned} \int \frac{x^3}{\sqrt{4 + x^2}} \, dx &= 8 \int (\tan \theta)^3 \sec \theta \, d\theta \\ &= 8 \int u^2 - 1 \, du = 8[u^3/3 - u] + C = 8[\frac{(\sec \theta)^3}{3} - \sec \theta] + C \end{aligned}$$

and it remains to write the answer in terms of the variable x. We draw the triangle and see that $\sec \theta = (4 + x^2)^{1/2}/2$ so that

$$\int \frac{x^3}{\sqrt{4 + x^2}} \, dx = 8 \left[\frac{(4 + x^2)^{3/2}}{24} - \frac{(4 + x^2)^{1/2}}{2} \right] + C = \frac{1}{3}(4 + x^2)^{1/2}[x^2 - 8] + C$$

We can check by differentiation

$$\begin{aligned} \frac{d}{dx} \frac{1}{3}(4 + x^2)^{1/2}[x^2 - 8] &= \frac{1}{3} \left\{ \frac{1}{2}(4 + x^2)^{-1/2}(2x)[x^2 - 8] + (4 + x^2)^{1/2}[2x] \right\} \\ &= \frac{1}{3}(4 + x^2)^{-1/2} \left\{ x^3 - 8x + (4 + x^2)2x \right\} = \frac{x^3}{(4 + x^2)^{1/2}} \end{aligned}$$

It checks

3. Set up an integral for the area of the *surface* generated by revolving the curve $y = \cos x$, $-\pi/2 \le x \le \pi/2$, about the x–axis. Do NOT evaluate the integral.

   We need $dy/dx = -\sin x$. The area is

$$\int_{-\pi/2}^{\pi/2} 2\pi \cos x \sqrt{1 + (-\sin x)^2}\, dx = \int_{-\pi/2}^{\pi/2} 2\pi \cos x \sqrt{1 + (\sin x)^2}\, dx$$

4. Compute the volume of the solid generated by revolving the region bounded by the curves $y = (x-2)^2$ and $y = x$ about the y–axis. (Suggestion: Sketch R.) (17)

   The region R is bounded below by the parabola $y = (x-2)^2$ which is the standard parabola $y = x^2$ shifted two units right and R is bounded above by the line $y = x$. The two curves intersect when $x = (x-2)^2$ or $0 = (x^2-4x+4)-x = x^2-5x+4 = (x-4)(x-1)$ that is when $x = 1$ and $x = 4$. The region is Type I (not Type II) because it is bounded above and below by simple curves and not on the left and right. Therefore we need to integrate in x and we should use cylindrical shells and the volume is

$$\begin{aligned}
\int_1^4 2\pi x(x - (x-2)^2)\, dx &= 2\pi \int_1^4 x(x - x^2 + 4x - 4)\, dx \\
&= 2\pi \int_1^4 -x^3 + 5x^2 - 4x\, dx \\
&= 2\pi \left[-\frac{1}{4}x^4 + \frac{5}{3}x^3 - 2x^2 \right]_1^4 = \frac{45\pi}{2}
\end{aligned}$$

5. The solid lies between planes perpendicular to the x–axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x–axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$. (17)

   The cross section are disks and so $A(x) = \pi r^2$ when r is a function of x. Indeed r is half the distance from $2 - x^2$ to x^2 which is $2 - 2x^2$: $r = 1 - x^2$. The volume is therefore

$$\begin{aligned}
\int_{-1}^1 \pi(1-x^2)^2\, dx &= \pi \int_{-1}^1 (1 - 2x^2 + x^4)\, dx \\
&= \pi \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \right]_{-1}^1 = \pi \left[1 - \frac{2}{3} + \frac{1}{5} - (-1 + \frac{2}{3} - \frac{1}{5} \right] = \frac{16}{15}
\end{aligned}$$

   The solid is also a solid of revolution about the axis $y = 1$ and so can be done by the disk method: $\int_{-1}^1 \pi(2 - x^2 - 1)^2 dx$.