 Test 1B, Math 1860 Section 022
2/13/14 Instr: Denis White; Ramesh Karki Name ____________________
There are five pages (two sided); the back page is for rough work.

1. Evaluate the integral.

(a) $\displaystyle\int_0^{\ln 2} \frac{e^{3x}}{2+e^{3x}}\,dx$ (Simplify your answer.) (11)

Substitute $u = 2 + e^{3x}$ so that $du = 3e^{3x}\,dx$ so that

$$\begin{aligned}\int_0^{\ln 2} \frac{e^{3x}}{2+e^{3x}}\,dx &= \frac{1}{3}\int_{x=0}^{x=\ln 2} \frac{1}{u}\,du \\ &= \frac{1}{3}\ln|u|\Big|_{x=0}^{x=\ln 2} = \frac{1}{3}\left[\ln|2+e^{3\ln 2}| - \ln|2+e^0|\right] = \frac{1}{3}\left[\ln 10 - \ln 3\right]\end{aligned}$$

(b) $\displaystyle\int t^2 e^{-t}\,dt$ (10)

Integrate by parts. Let $u = t^2$ so that $dv = e^{-t}$ and $du = 2t\,dt$ and $v = -e^{-t}$. Plugging into the formula ($\int u\,dv = uv - \int v\,du$), we have

$$\int t^2 e^{-t}\,dt = t^2(-e^{-t}) - \int 2t(-e^{-t})\,dt$$

Integrate by parts again. Let $u = t$ so that $dv = e^{-t}\,dt$, $du = dt$ and $v = -e^{-t}$. Therefore

$$\begin{aligned}\int t^2 e^{2t}\,dt &= -t^2e^{-t} + 2\left[t(-e^{-t}) - \int -e^{-t}\,dt\right] \\ &= -t^2e^{-t} - 2te^{-t} - 2e^{-t} + C = -e^{-t}\left[t^2+2t+2\right]\end{aligned}$$

and we can check by differentiation: by the product rule

$$\frac{d}{dt}\left[-e^{-t}\left[t^2+2t+2\right]\right] = -e^{-t}\left[2t+2\right] + e^{-t}\left[t^2+2t+2\right] = t^2e^{2t}$$

(c) $\displaystyle\int (\sin 2x)^2(\cos 2x)^3\,dx$ (10)

There is an odd power of $\cos 2x$ and so we substitute $u = \sin 2x$ so that $du = 2\cos 2x\,dx$. We further use the trig identity $(\cos 2x)^2 = 1-(\sin 2x)^2 = 1-u^2$ so that

$$\begin{aligned}\int (\sin 2x)^2(\cos 2x)^3\,dt &= \frac{1}{2}\int u^2(1-u^2)\,du \\ &= \frac{1}{2}\int u^2 - u^4\,du \\ &= \frac{1}{2}\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right] = \frac{1}{6}(\sin 2x)^3 - \frac{1}{10}(\sin 3x)^5 + C\end{aligned}$$

Check by differentiation.

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{6}(\sin 2x)^3 - \frac{1}{10}(\sin 2x)^5\right] &= \frac{3}{6}(\sin 2x)^2(\cos 2x)2 - \frac{5}{10}(\sin 2x)^4(\cos 2x)2 \\ &= (\sin 2x)^2 \cos 2x\left[1-(\sin 2x)^2\right] \\ &= (\sin 2x)^2(\cos 2x)^3\end{aligned}$$

(10) (d) $\int (\cot\theta)^2\, dx$

Here we use the identity $(\cot\theta)^2 = (\csc\theta)^2 - 1$.

$$\int (\cot\theta)^2\, dx = \int (\csc\theta)^2 - 1\, dx = -\cot 2\theta - \theta + C$$

and we can check by differentiation.

$$\frac{d}{d\theta}\left[\frac{1}{2}\tan 2\theta - \theta\right] = \frac{1}{2}(\sec 2\theta)^2(2) - 1 = (\tan 2\theta)^2$$

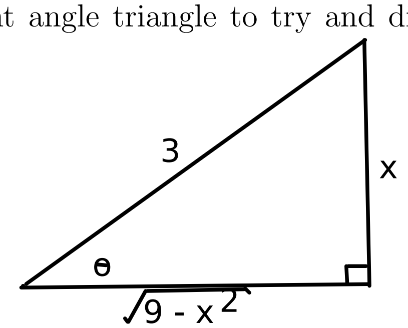
by that same trig identity.

(13) (e) $\int \frac{1}{(9-x^2)^{3/2}}\, dx$

A trig substitution $x = 3\sin\theta$ is appropriate. Then $dx = 3\cos\theta d\theta$ and $(9-x^2)^{1/2} = 3\cos\theta$

$$\begin{aligned}\int \frac{1}{(9-x^2)^{3/2}}\, dx &= \int \frac{1}{(3\cos\theta)^3} 3\cos\theta\, d\theta \\ &= \frac{1}{9}\int (\sec\theta)^2\, d\theta = \frac{1}{9}\tan\theta + C\end{aligned}$$

We now draw a right angle triangle to try and discover what $\tan\theta$ is in



terms of x. It is opposite divided by adjacent or x divided by $\sqrt{9-x^2}$.

$$\int \frac{1}{(9-x^2)^{3/2}}\, dx = \frac{1}{9}\frac{x}{\sqrt{9-x^2}} + C$$

and we can check this by differentiation: By the quotient rule

$$\begin{aligned}\frac{d}{dx}\frac{1}{9}\frac{x}{\sqrt{9-x^2}} &= \frac{1}{9}\frac{(9-x^2)^{1/2} - x(1/2)(9-x^2)^{-1/2}(-2x)}{((9-x^2)^{1/2})^2} \\ &= \frac{1}{9}\frac{(9-x^2)^{1/2}(9-x^2)^{1/2} + x^2}{(9-x^2)^{3/2}} = \frac{1}{(9-x^2)^{3/2}}\end{aligned}$$

It checks.

2. Find the length of the curve $y = (1/3)(x^2+2)^{3/2}$, from $x = 1$ to $x = 2$. (13)

We need $y' = (1/3)(3/2)(x^2+2)^{1/2}2x = x(x^2+2)^{1/2}$. The length of the curve is

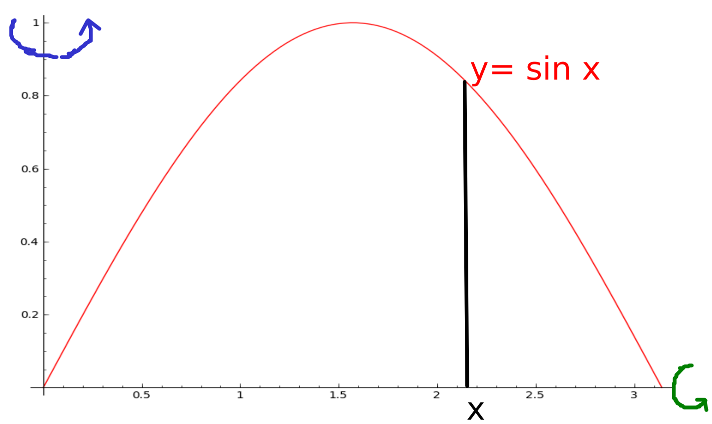
$$\begin{aligned}\int_1^2 \sqrt{1+(y')^2}\,dx &= \int_1^2 \sqrt{1+x^2(x^2+2)}\,dx \\ &= \int_1^2 \sqrt{1+2x^2+x^4}\,dx \\ &= \int_1^2 1+x^2\,dx = \left[x+\frac{1}{3}x^3\right]_1^2 = 2+\frac{8}{3}-(1+\frac{1}{3}) = \frac{10}{3}\end{aligned}$$

3. Set up an integral for the area of the *surface* generated by revolving the curve $x = y^2$, $1 \le y \le 3$, about the y–axis. Do NOT evaluate the integral. (10)

The surface area of a surface of rotation about the y–axis is

$$\int_1^3 2\pi x\sqrt{1+(x')^2}\,dy = 2\pi\int_1^3 y^2\sqrt{1+(2y)^2}\,dy = 2\pi\int_1^3 y^2\sqrt{1+4y^2}\,dy$$

4. Consider the region bounded by the curves $y = \sin x$, $0 \le x \le \pi$ and $y = 0$. Find the volumes of the solids generated when that region is rotated about the specified axes. (Suggestion: Sketch R.)



(23)

(a) about the x–axis.

Sketch. The volume is, by the method of washers.

$$\int_0^\pi \pi(\sin x)^2\,dx = \frac{\pi}{2}\int_0^\pi 1-\cos 2x\,dx = \frac{\pi}{2}\left[x-\frac{1}{2}\sin 2x\right]_0^\pi = \pi-\frac{1}{2}\sin 2\pi-0 = \pi$$

(b) about the y–axis. (Set up an integral for the volume but do NOT evaluate.)

The volume is, by the method of cylindrical shells

$$\int_0^\pi 2\pi x\sin x\,dx$$