 Test 1A, Math 1860 Section 021
2/13/14 Solutions Name ____________________

There are five pages (two sided); the back page is for rough work.

1. Evaluate the integral.

(a) $\displaystyle\int_0^{\ln 2} \frac{e^{-x}}{2+e^{-x}}\,dx$ (Simplify your answer.) (11)

Substitute $u = 2+e^{-x}$ so that $du = -e^{-x}\,dx$ so that

$$\begin{aligned}\int_0^{\ln 2} \frac{e^{-x}}{2+e^{-x}}\,dx &= -\int_{x=0}^{x=\ln 2} \frac{1}{u}\,du \\ &= -\ln|u|\big|_{x=0}^{x=\ln 2} \\ &= -\ln|2+e^{-x}|\big|_0^{\ln 2} \\ &= -\ln|2+e^{-\ln 2}| + \ln|2+e^0| = -\ln|2+1/2| + \ln|2+1| = \ln\frac{6}{5}\end{aligned}$$

and this can be checked by differentiation:

$$\frac{d}{dx}[-\ln|2+e^{-x}|] = -\frac{1}{2+e^{-x}}(e^{-x}(-1)) = \frac{e^{-x}}{2+e^{-x}}$$

(b) $\displaystyle\int t^2 e^{2t}\,dt$ (10)

Integrate by parts. Let $u = t^2$ so that $dv = e^{2t}$ and $du = 2t\,dt$ and $v = (1/2)e^{2t}$. Plugging into the formula ($\int u\,dv = uv - \int v\,du$), we have

$$\int t^2 e^{2t}\,dt = \frac{t^2}{2}e^{2t} - \int te^{2t}\,dt$$

Integrate by parts again. Let $u = t$ so that $dv = e^{2t}\,dt$, $du = dt$ and $v = (1/2)e^{2t}$. Therefore

$$\int t^2 e^{2t}\,dt = \frac{t^2}{2}e^{2t} - \left[\frac{t}{2}e^{2t} - \frac{1}{2}\int e^{2t}\,dt\right] = \frac{t^2}{2}e^{2t} - \frac{t}{2}e^{2t} + \frac{1}{4}e^{2t} + C = e^{2t}\left[\frac{t^2}{2} - \frac{t}{2} + \frac{1}{4}\right]$$

and we can check by differentiation: by the product rule

$$\frac{d}{dt}\left[e^{2t}\left[\frac{t^2}{2} - \frac{t}{2} + \frac{1}{4}\right]\right] = e^{2t}\left[t - \frac{1}{2}\right] + e^{2t}2\left[\frac{t^2}{2} - \frac{t}{2} + \frac{1}{4}\right] = t^2e^{2t} \text{ It checks.}$$

(c) $\displaystyle\int (\cos 3x)^2(\sin 3x)^3\,dx$ (10)

There is an odd power of $\sin 3x$ and so we substitute $u = \cos 3x$ so that $du = -3\sin 3x\,dx$. Further we use the identity $(\sin 3x)^2 = 1-(\cos 3x)^2 = 1-u^2$ so that

$$\begin{aligned}\int (\cos x)^2(\sin x)^3\,dx &= -\frac{1}{3}\int u^2(1-u^2)\,du \\ &= -\frac{1}{3}\int u^2 - u^4\,du \\ &= -\frac{1}{9}u^3 + \frac{1}{15}u^5 + C = -\frac{1}{9}(\cos 3x)^3 + \frac{1}{15}(\cos 3x)^5 + C\end{aligned}$$

Check by differentiation.

$$
\begin{aligned}
&\frac{d}{dx}\left[-\frac{1}{9}(\cos 3x)^3 + \frac{1}{15}(\cos 3x)^5\right] \\
&= -\frac{1}{9}\,3(\cos 3x)^2(-3\sin 3x) + \frac{1}{15}\,5(\cos 3x)^4(-3\sin 3x) \\
&= (\cos 3x)^2 \sin 3x\left[1-(\cos 3x)^2\right] = (\cos 3x)^2(\sin 3x)^3
\end{aligned}
$$

(10) (d) $\int (\tan 2\theta)^2\,dx$

Here we use the identity $(\tan 2\theta)^2 = (\sec 2\theta)^2 - 1$.

$$\int (\tan 2\theta)^2\,dx = \int (\sec 2\theta)^2 - 1\,dx = \frac{1}{2}\tan 2\theta - \theta + C$$

and we can check by differentiation.

$$\frac{d}{d\theta}\left[\frac{1}{2}\tan 2\theta - \theta\right] = \frac{1}{2}(\sec 2\theta)^2(2) - 1 = (\tan 2\theta)^2$$

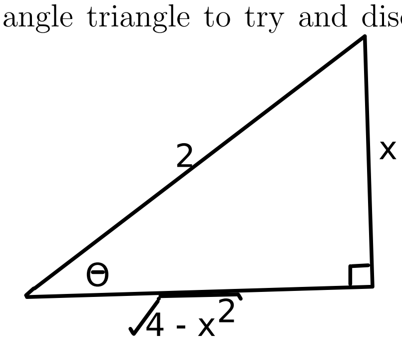
by that same trig identity.

(13) (e) $\int \frac{1}{(4-x^2)^{3/2}}\,dx$

A trig substitution $x = 2\sin\theta$ is appropriate. Then $dx = 2\cos\theta d\theta$ and $(4-x^2)^{1/2} = 2\cos\theta$

$$
\begin{aligned}
\int \frac{1}{(4-x^2)^{3/2}}\,dx &= \int \frac{1}{(2\cos\theta)^3} 2\cos\theta\,d\theta \\
&= \frac{1}{4}\int (\sec\theta)^2\,d\theta = \frac{1}{4}\tan\theta + C
\end{aligned}
$$

We now draw a right angle triangle to try and discover what $\tan\theta$ is in



terms of x. It is opposite divided by adjacent or x divided by $\sqrt{4-x^2}$.

$$\int \frac{1}{(4-x^2)^{3/2}}\,dx = \frac{1}{4}\frac{x}{\sqrt{4-x^2}} + C$$

and we can check this by differentiation:

$$
\begin{aligned}
\frac{d}{dx}\frac{1}{4}\frac{x}{\sqrt{4-x^2}} &= \frac{1}{4}\frac{(4-x^2)^{1/2} - x(1/2)(4-x^2)^{-1/2}(-2x)}{((4-x^2)^{1/2})^2} \\
&= \frac{1}{4}\frac{(4-x^2)^{1/2}(4-x^2)^{1/2} + x^2}{(4-x^2)^{3/2}} = \frac{1}{(4-x^2)^{3/2}}
\end{aligned}
$$

It checks.

2. Find the length of the curve $x = (1/3)(y^2+2)^{3/2}$, from $y = 0$ to $y = 3$. (13)

We need $x' = (1/3)(3/2)(y^2+2)^{1/2}2y = y(y^2+2)^{1/2}$. The length of the curve is

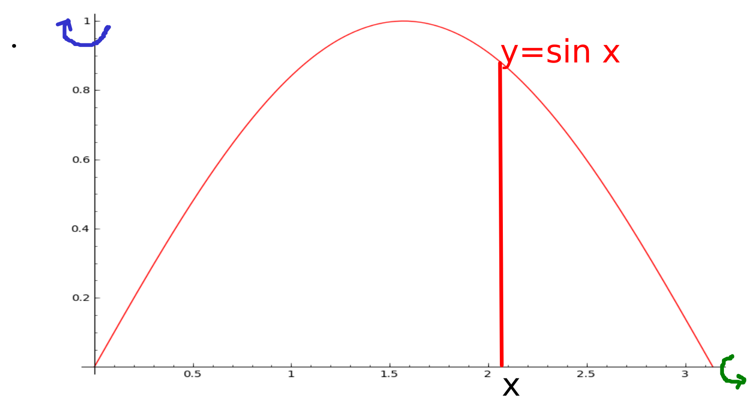
$$\int_0^3 \sqrt{1+(x')^2}\,dy = \int_0^3 \sqrt{1+y^2(y^2+2)}\,dy$$
$$= \int_0^3 \sqrt{1+2y^2+y^4)}\,dy = \int_0^3 1+y^2\,dy = \left[y+\frac{1}{3}y^3\right]_0^3 = 3+\frac{1}{3}3^3 = 12$$

3. Set up an integral for the area of the *surface* generated by revolving the curve $y = x^{5/2}$, $1 \le x \le 3$, about the x–axis. Do NOT evaluate the integral. (10)

The surface area of a surface of rotation about the x–axis is

$$\int_1^3 2\pi y\sqrt{1+(y')^2}\,dx = 2\pi\int_1^3 x^{5/2}\sqrt{1+(5x^{3/2}/2)^2}\,dx = 2\pi\int_1^3 \sqrt{x^5(1+\frac{25}{4}x^3)}\,dx$$

4. Consider the region bounded by the curves $y = \sin x$, $0 \le x \le \pi$ and $y = 0$. Find the volumes of the solids generated when that region is rotated about the specified axes. (Suggestion: Sketch R.) (23)



(a) about the x–axis. Sketch. The volume is, by the method of washers.

$$\int_0^\pi \pi(\sin x)^2\,dx = \frac{\pi}{2}\int_0^\pi 1-\cos 2x\,dx = \frac{\pi}{2}\left[x-\frac{1}{2}\sin 2x\right]_0^\pi = \pi - \frac{1}{2}\sin 2\pi - 0 = \pi$$

(b) about the y–axis. (Set up an integral for the volume but do NOT evaluate.) The volume is, by the method of cylindrical shells

$$\int_0^\pi 2\pi x\sin x\,dx$$