Final Review, Math 1860
Thomas Calculus Early Transcendentals, 12 ed

6. Applications of Integration:

5.6 (Review Section 5.6) Area between curves $y = f(x)$ and $y = g(x)$, $a \le x \le b$ is $\int_a^b |f(x) - g(x)|\,dx$ and similarly for $x = f(y)$ and $x = g(y)$.

6.1 Volumes by slices: $\int_a^b A(x)\,dx$ where $A(x)$ is the area of the cross section perpendicular to the x-axis. Special case is the solid of rotation obtained by rotating the region between curves $y = f(x)$ and $y = g(x)$, $a \le x \le b$ about the x-axis:

$$\int_a^b \pi|f(x)^2 - g(x)^2|\,dx.$$

What happens in the case the axis of rotation is $y = 3$? There is an analogous formula obtained by interchanging x and y. Example: The region between the curves $x = y^2$ and $y = x - 2$ is rotated about the y-axis. Find the volume of the solid generated.

6.2 The method of cylindrical shells: the region between curves $y = f(x)$ and $y = g(x)$, $a \le x \le b$ about the y-axis: $\int_a^b 2\pi x[f(x) - g(x)]dx$. What happens when the axis of rotation is $x = -2$? Again interchanging x and y gives an analogous formula. Example: Rotate the region in the previous example around the x-axis.

6.3 Arc Length. The curve $y = f(x)$, $a \le x \le b$ has length

$$\int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

6.4 Area of Surfaces of Rotation: The curve $y = f(x)$, $a \le x \le b$ is totated about the x axis. Find the area of the surface.

$$2\pi \int_a^b |f(x)|\sqrt{1 + (f'(x))^2}\,dx$$

7. Transcendental Functions:

7.1 The **Natural Logarithm** $\ln x = \int_1^x (1/t)\,dt$ has has the property $\ln(xy) = \ln x + \ln y$ etc. $(d/dx)\ln u = (1/u)du/dx$. What is $\int \tan x\,dx$?

7.1 e^x, the **Natural Exponential** is the inverse of $\ln x$: $e^{\ln x} = x = \ln(e^x)$. $(d/dx)e^u = e^u du/dx$ and we saw $exp(x) = e^x$ if x is rational and so for all

7.1 General logs and exponentials. Differentiate $y = 3^{\sec x}$. or $y = \log_2 \sqrt{x}$.

7.3 Hyperbolic Functions: $\cosh x = \dfrac{e^x + e^{-x}}{2}$; $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\tanh x = \dfrac{\sinh x}{\cosh x}$.

8. Techniques of Integration:

8.1 Integration by parts: $\int u\,dv = uv - \int v\,du$. On p 407, what is u and dv in questions 1-24?

8.2 Integration of powers of Trig functions **Identities:**

(a) $(\sin x)^2 + (\cos x)^2 = 1$

(b) $(\cos x)^2 = \dfrac{1}{2}(1 + \cos 2x)$

(c) $(\sin x)^2 = \dfrac{1}{2}(1 - \cos 2x)$

(d) $(\sec x)^2 = (\tan x)^2 + 1$

(e) $(\csc x)^2 = (\cot x)^2 + 1$

(f) $\sin 2x = 2 \sin x \cos x$

(g) $\cos 2x = (\cos x)^2 - (\sin x)^2$

$$\int (\sin x)^m (\cos x)^n \, dx$$

Cases: m is odd ($u = \cos x$); n is odd ($u = \sin x$); m and n even

$$(\cos x)^2 = \frac{1}{2}(1 + \cos 2x) \quad (\sin x)^2 = \frac{1}{2}(1 - \cos 2x)$$

8.2 Integration of powers of Trig functions

$$\int (\tan x)^m (\sec x)^n \, dx$$

If n is even substitute $u = \tan x$. If m is odd then substitute $u = \sec x$. Also

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

Also

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

8.3 Trig Substitution

For Integrals Involving	Substitute	Use the Identity
$\sqrt{a^2 - x^2}$	$x = a \sin\theta,\ dx = a\cos\theta\, d\theta$	$\sqrt{a^2 - x^2} = a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\ dx = a(\sec\theta)^2\, d\theta$	$\sqrt{a^2 + x^2} = a\sec\theta$

8.4 Partial Fractions. 1. Divide; 2. Factor divisor 3. Expand by partial fractions 4. Solve for coefficients 5. Integrate.

8 Evaluate

$$\int x \arctan x \, dx =$$

$$\int (\cos x)(\sin x)^6 \, dx =$$

$$\int (\cos x)^4 \, dx =$$

$$\int (\tan x)^3 \, dx =$$

$$\int \frac{\sqrt{1-v^2}}{v^2} \, dv =$$

$$\int \frac{1+x^2}{(1+x)^3} \, dx$$

8.7 Improper integrals. Type I and II. $\displaystyle\int_1^\infty \frac{1}{x^p} \, dx < \infty$ if and only if $p > 1$.

1. Comparison Test. Assume $0 \leq f(x) \leq g(x)$. Then $\int_a^\infty g(x) \, dx < \infty$ implies $\int_a^\infty f(x) \, dx < \infty$ OR $\int_a^\infty f(x) \, dx = \infty$ implies $\int_a^\infty g(x) \, dx = \infty$

   10. Series:

10.1 Sequences

(a) $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n} = 0$

(b) $\displaystyle\lim_{n\to\infty} n^{1/n} = 1$

(c) $\displaystyle\lim_{n\to\infty} x^{1/n} = 1$

(d) $\displaystyle\lim_{n\to\infty} x^n = 0$ if $|x| < 1$.

(e) $\displaystyle\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

(f) $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = 0$

2. Series.

   10.2 Geometric series $a + ar + ar^2 + ar^3 + \ldots = a/(1-r)$.

   10.2 Telescoping series.

   10.2 nth term test for divergence. $\lim_n a_n \neq 0$

   10.3 Integral Test

   10.3 p-series $\sum_n^\infty 1/n^p$ converges iff $p > 1$.

   10.4 Comparison Test $\sum_n^\infty 1/(n^2+n)$

10.4 Limit Comparison Test.$\sum_n^\infty 1/(n^2 - n)$

10.6 Conditional or absolute convergence?

10.5 Alternating Series Test. Is $\sum_n^\infty (-1)^{n-1}/\sqrt{n}$ conditionally convergent? AST is only for conditional convergence.

10.6 Ratio Test. $\lim_n |a_{n+1}/a_n|$

10.6 Root Test. $\lim_n |a_n|^{1/n}$

10.7 Power series $\sum_{n=0}^\infty c_n(x-a)^n$. Radius of convergence and interval of convergence. Ratio or root test. Check end points?

10.8 Integration and differentiation of power series: $f(x) = \sum_{n=0}^\infty c_n(x-a)^n$

$$\begin{aligned} f'(x) &= \sum_{n=1}^\infty nc_n(x-a)^{n-1} \\ \int f(x)\,dx &= C + \sum_{n=0}^\infty \frac{c_n}{n+1}(x-a)^{n+1} \end{aligned}$$

All three power series have the same radius of convergence.

10.8 The geometric series and power series. For example, $1/x = 1/(1+(x-1)) = 1-(x-1)+(x-1)^2-(x-1)^3+\ldots$ (Here $r = -(x-1)$ and $a = 1$. Convergence if $|r| = |x-1| < 1$.

10.9 Taylor series and Maclaurin series ($a = 0$)

$$\sum_{n=0}^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and Taylor polynomials, $T_N(x)$ of degree N is $P_N(x) = \sum_{n=0}^N \frac{f^{(n)}(a)}{n!}(x-a)^n$

10.9 Common Maclaurin series:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \ldots \frac{x^n}{n!} + \ldots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots \end{aligned}$$

10.10 Integration of power series like that of e^{x^2}

Parametric and Polar Curves.

11.1 Parametric Curves. Graphing

11.2 Parametric Curves and Calculus. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$. Tangent line to a parametric curve.

Length (compare to Section 6.3) $\displaystyle\int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

11.3 Polar Coordinates

11.4 Polar Coordinates. Graphing. Circles (r =constant. $r = a\cos\theta$ $r = a\sin\theta$) and cardioids ($r = a(1 \pm \cos\theta)$ or $r = a(1 \pm \sin\theta)$) and the flowers like $r^2 = \sin 3\theta$.

11.5 Area in polar coordinates $\displaystyle\int_\alpha^\beta \frac{1}{2} f(\theta)^2\, d\theta$

$\mathbb{R}^3$

12.1 Distance between points.

12.1 Equation of a sphere centered at (4,-1,3) and radius 8.

12.2 Vectors. Length and direction. Unit vectors. Force, displacement and velocity.

12.3 Dot product. Algebraic and geometric difinitions. Angle between vectors.

12.3 Component of $\vec{b}$ along $\vec{a}$: $\text{comp}_{\vec{a}}\vec{b} = (\vec{a} \cdot \vec{b})/|\vec{a}|$. Projection of $\vec{b}$ along $\vec{a}$: $\text{proj}_{\vec{a}}\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\vec{a}$.

12.4 Cross product. Algebraic and geometric definitions. $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$ and $\vec{b}$ and by the right hand rule. Length is the area of the parallelogram.

12.4 Triple vector product $\vec{a} \cdot (\vec{b} \times \vec{c})$. Take absolute value of this real number to get the volume of the parallelepiped determined by the 3 vectors.

12.4 $\vec{a} \cdot \vec{b}$ is a real number; $\vec{a} \times \vec{b}$ is a vector.

12.5 Equation of a plane through three points P Q and R. The normal is $\vec{PQ} \times \vec{PR}$.

12.5 Parametric equation of the line through $P(a, b, c)$ and Q is $\langle x, y, z\rangle = \langle a, b, c\rangle + t\vec{PQ}$. Here $\vec{PQ} = \langle d_1, d_2, d_3\rangle$ is the direction vector of the line and we have $x = a + td_1$, $y = b + td_2$ and $z = c + td_3$ if we write out the components.

12.5 The symmetric equations of the same line are

$$\frac{x - a}{d_1} = \frac{y - b}{d_2} = \frac{z - c}{d_3}$$