Two Pages! **Quiz 7B**, Math 1860-022
3/20/14 Solutions Name ________________

(4) 1. Use the Comparison Test to determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$$

We compare to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ which is a p series with $p = 3/2 > 1$ and so it converges. We also see that $0 \leq \cos^2 n \leq 1$ so that

$$\frac{\cos^2 n}{n^{3/2}} \leq \frac{1}{n^{3/2}}$$

and so the Comparison Test applies and we see that the given series is smaller than a convergent p series and so it must also converge.

(4) 2. Use the Limit Comparison Test to determine if the series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)}$$

We want to compare to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ which is a p series with $p = 1$ (harmonic series) and since $p \leq 1$ it diverges. Try Limit Comparison

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{n(n+1)}{(n^2+1)(n-1)}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n^2(n+1)}{(n^2+1)(n-1)} = 1$$

where the limit follows by l'Hôpital's Rule (or factor n^3 out of top and bottom). Since the limit is $0 < 1 < \infty$, the Limit Comparison Test says that the two series both converge or both diverge but we know that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and so the given series diverges.

(4) 3. Does the series converge or diverge? Use any method but give reasons for your answer.

$$\sum_{n=1}^{\infty} \frac{1-n}{n2^n}$$

There are many ways to do this one. One simple way is to write it as the difference of two series

$$\sum_{n=1}^{\infty} \frac{1-n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n2^n} - \sum_{n=1}^{\infty} \frac{1}{2^n}$$

The second series is a geometric series with $r = 1/2$ and is convergent because $|r| = 1/2 < 1$. The first series is smaller than the second

$$\frac{1}{n2^n} \leq \frac{1}{2^n}$$

and so it too must converge (by the Direct Comparison Test). This shows the given series converges. Alternatively one could observe that

$$-\frac{1}{2^n} \leq \frac{1-n}{n2^n} \leq 0$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges (geometric series with $r = 1/2$) it follows by the Comparison Test that the given series converges. A third possibility is to apply the ratio test.

(4 ea)

4. Does the *series* converge or diverge? Give reasons for your answer. The ratio or root test might help

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(3n)^n}$

   This is a good candidate for the root test because there are powers of n. Consider therefore

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{4}{3n} = 0$$

   and the limit is smaller than 1 and so the series converges absolutely by the root test.

   (b) $\displaystyle\sum_{n=1}^{\infty} n!e^{-n}$

   This is a good candidate for the ratio test because there are lots of multiplications between terms. Consider therefore

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!e^{-(n+1)}}{n!e^{-n}} = \lim_{n\to\infty} \frac{e^n}{e^{n+1}} \frac{(n+1)!}{n!} = \lim_{n\to\infty} \frac{n+1}{e} = \infty$$

   (where we note $e^{n+1} = e(e^n)$ and $(n+1)! = (n+1)n!$.) Since the limit is larger than 1 the series diverges by the ratio test.