Two Pages! **Quiz 6C**, Math 1860-022
3/13/14 Solutions Name ________________

1. Write out the first few terms of the *series* to show how the series starts. Then find
(8) the sum of the series.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{7}{4^n} = 7 + \frac{7}{4} + \frac{7}{4^2} + \dots$
   Here $a = 7$ and $r = 1/4$ and since $|r| < 1$ the series converges to $a/(1-r)$

$$\sum_{n=0}^{\infty} \frac{7}{4^n} = \frac{7}{1-1/4} = \frac{28}{3}$$

   (b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right) = 1 + 1 + \frac{1}{2} - \frac{1}{5} + \frac{1}{4} + \frac{1}{25} + \dots$
   This is the sum of two geometric series $1+1/2+1/4+\dots$ and $1-1/5+1/25+\dots$. For the first $a = 1$ and $r = 1/2$ and $|r| < 1$ so that the first series converges; for the second $a = 1$ and $r = -1/5$ and again $|r| < 1$ and so both series converge and we have

$$\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right) = \frac{1}{1-1/2} - \frac{1}{1-(-1/5)} = \frac{17}{6}$$

2. For the geometric series below, determine a and r and find the sum of the series. Then express the inequality $|r| < 1$ in terms of x and find the values of x for which
(5) the inequality holds and the series converges.

$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n = 1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \dots$$

   Here $a = 1$ and $r = -(x-3)/2$ and so

$$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n = \frac{1}{1-(-(x-3)/2)} = \frac{2}{2+(x-3)} = \frac{2}{x-1}$$

   provided $|r| < 1$ which means $|-(x-3)/2| < 1$ or $|x-3| < 2$ or $1 < x < 5$

3. Does the *series* converge or diverge? Give reasons for your answer. If the series
(2) converges then find its limit.

$$\sum_{n=0}^{\infty} \frac{n!}{1000^n}$$

   This series diverges by the nth term test for divergence because the terms do not converge to zero. Indeed

$$\lim_{n\to\infty} \frac{n!}{1000^n} = \infty$$

   (See page 556 of the text.)

4. Apply the Integral Test to determine if the *series* converges or diverges.

$$\sum_{n=2}^{\infty} \frac{n}{n^2+1}$$

(5)

Consider $f(x) = \dfrac{x}{x^2+1}$, for $x \geq 1$. Observe $f(x) \geq 0$ and $f(x)$ is decreasing because $f'(x) = (1-x^2)/(x^2+1)$ and that is negative if $x > 1$. Evaluate the improper integral

$$\int_1^{\infty} \frac{x}{x^2+1}\, dx = \lim_{b\to\infty} \int_1^b \frac{x}{x^2+1}\, dx = \lim_{b\to\infty} \frac{1}{2} \int_2^{b^2+1} \frac{1}{u}\, du$$

where we have substituted $u = x^2+1$ so that $du = 2x\, dx$. Therefore

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx = \frac{1}{2} \lim_{b\to\infty} [\ln u]_2^{b^2+1} = \frac{1}{2} \lim_{b\to\infty} \ln(b^2+1) - \ln 2 = \infty$$

Thus the improper integral diverges and so the series must diverge also by the Integral Test:

$$\sum_{n=1}^{\infty} \frac{n}{n^2+4} \text{ diverges}$$