Two Pages! **Quiz 6A**, Math 1860-021
3/13/14 Solutions Name ____________________

1. Write out the first few terms of the *series* to show how the series starts. Then find
(8) the sum of the series.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} = 1 - \frac{1}{4} + \frac{1}{4^2} + \ldots$
   Here $a = 1$ and $r = -1/4$ and since $|r| < 1$ the series converges to $a/(1-r)$
   $$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} = \frac{1}{1-(-1/4)} = \frac{4}{5}$$

   (b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right) = 5 - 1 + \frac{5}{2} - \frac{1}{3} + \frac{5}{4} - \frac{1}{9} + \ldots$
   This is the difference of two geometric series $5 + 5/2 + 5/4 + \ldots$ and $1 + 1/3 + 1/9 + \ldots$. For the first $a = 5$ and $r = 1/2$ and $|r| < 1$ so that the first series converges; for the second $a = 1$ and $r = 1/3$ and again $|r| < 1$ and so both series converge and we have
   $$\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right) = \frac{5}{1-1/2} - \frac{1}{1-1/3} = \frac{17}{2}$$

2. For the geometric series below, determine a and r and find the sum of the series. Then express the inequality $|r| < 1$ in terms of x and find the values of x for which
(5) the inequality holds and the series converges.
   $$\sum_{n=0}^{\infty} (-1)^n (x+1)^n = 1 - (x+1) + (x+1)^2 + \ldots$$
   Here $a = 1$ and $r = -(x+1)$ and so
   $$\sum_{n=0}^{\infty} (-1)^n (x+1)^n = \frac{1}{1-(-(x+1))} = \frac{1}{2+x} = \frac{1}{2+x}$$
   provided $|r| < 1$ which means $|(x-1)/2| < 1$ or $|x-1| < 2$ or $-1 < x < 3$

3. Does the *series* converge or diverge? Give reasons for your answer. If the series
(2) converges then find its limit.
   $$\sum_{n=1}^{\infty} \frac{2^n}{n+1}$$
   The series diverges by the nth term test for divergence because the terms do not converge to 0: Indeed
   $$\lim_{n\to\infty} \frac{2^n}{n+1} \left(= \frac{\infty}{\infty}\right) = \lim_{n\to\infty} \frac{(\ln 2)2^n}{1} = \infty$$
   by l'Hospital's rule.

4. Apply the Integral Test to determine if the *series* converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n}{n^2+4}$$

(5)

Consider $f(x) = \dfrac{x}{x^2+1}$, for $x \geq 1$. Observe $f(x) \geq 0$ and $f(x)$ is decreasing because $f'(x) = (1-x^2)/(x^2+1)$ and that is negative if $x > 1$. Evaluate the improper integral

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^b \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \frac{1}{2} \int_2^{b^2+1} \frac{1}{u}\,du$$

where we have substituted $u = x^2+1$ so that $du = 2x\,dx$. Therefore

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \frac{1}{2} \lim_{b\to\infty} [\ln u]_2^{b^2+1} = \frac{1}{2} \lim_{b\to\infty} \ln(b^2+1) - \ln 2 = \infty$$

Thus the improper integral diverges and so the series must diverge also by the Integral Test:

$$\sum_{n=1}^{\infty} \frac{n}{n^2+4} \text{ diverges}$$