Two Pages! **Quiz 5B**, Math 1860-022
2/27/14 Solutions Name ____________________

1. Express the integrand as a sum of partial fractions and evaluate the integral.

(12) $\displaystyle\int_0^1 \frac{1}{(x+1)(x^2+1)}\,dx$

The integrand $1/(x+1)(x^2+1)$ already has the degree on bottom 3 higher than that on top (0) and moreover the bottom is already factored. Therefore we can already apply the partial fractions expansion which is of the form

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

for some constants A, B and C. To solve for these constants we multiply by $(x+1)(x^2+1)$ and collect like coefficients.

$$1 = A(x^2+1) + (Bx+C)(x+1) = (A+B)x^2 + (B+C)x + A + C$$

Equate like coefficients:
$$\begin{array}{ccccccc} A & + & B & & & = & 0 \\ & & B & + & C & = & 0 \\ A & & & + & C & = & 1 \end{array}$$

If we subtract equation 2 from 1 we get $A - C = 0$ so $A = C$. Substituting into the third equation, gives $A = C = 1/2$ so that $B = -1/2$. Therefore

$$\frac{1}{(x+1)(x^2+1)} = \frac{1/2}{x+1} + \frac{-(1/2)x + (1/2)}{x^2+1} = \frac{1}{2}\frac{1}{x+1} - \frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\frac{1}{x^2+1}$$

Integrate each of the three terms separately

$$\int_0^1 \frac{1}{(x+1)(x^2+1)}\,dx = \frac{1}{2}\int_0^1 \frac{1}{x+1}\,dx - \frac{1}{2}\int_0^1 \frac{x}{x^2+1}\,dx + \frac{1}{2}\int_0^1 \frac{1}{x^2+1}\,dx$$

In the middle integral we substitute $u = x^2+1$ so that $du = 2x\,dx$ and the endpoints of integration become $u(0) = 1$ and $u(1) = 2$

$$\begin{aligned} \int_0^1 \frac{1}{(x+1)(x^2+1)}\,dx &= \frac{1}{2}\ln|x+1|\Big|_0^1 - \frac{1}{4}\int_1^2 \frac{1}{u}\,du + \frac{1}{2}\arctan x\Big|_0^1 \\ &= \frac{1}{2}[\ln 2 - \ln 1] - \frac{1}{4}\ln u_1^2 + \frac{1}{2}[\arctan 1 - \arctan 0] \\ &= \frac{1}{2}\ln 2 - \frac{1}{4}[\ln 2 - \ln 1] + \frac{1}{2}\frac{\pi}{4} = \frac{1}{4}\ln 2 + \frac{\pi}{8} \end{aligned}$$

(8) 2. Evaluate the integral. $\displaystyle\int_2^\infty \frac{2}{t^2-1}\,dt$

Again this integral is by partial fractions. Division is not required. The bottom factors as $t^2 - 1 = (t+1)(t-1)$ and so the partial fractions expansion is

$$\frac{2}{t^2-1} = \frac{2}{(t+1)(t-1)} = \frac{A}{t+1} + \frac{B}{t-1}$$

To find the actual constants A and B, multiply both sides by $(t+1)(t-1)$ and collect up like coefficients.

$$2 = A(t-1) + B(t+1) = (A+B)t + B - A$$

Equating coefficients gives $A + B = 0$ and $B - A = 2$ so that $B = 1$ and $A = -1$. Therefore

$$\frac{2}{t^2-1} = \frac{2}{(t+1)(t-1)} = -\frac{1}{t+1} + \frac{1}{t-1}$$

Integrate.

$$\begin{aligned}
\int_2^\infty \frac{2}{t^2-1}\,dt &= \lim_{s\to\infty} \int_2^s -\frac{1}{t+1} + \frac{1}{t-1}\,dt \\
&= \lim_{s\to\infty} \left[-\ln|t+1| + \ln|t-1|\right]_2^s \\
&= \lim_{s\to\infty} \ln\left|\frac{t-1}{t+1}\right|_2^s \\
&= \lim_{s\to\infty} \ln\left[\frac{s-1}{s+1}\right] - \ln\frac{1}{3} \\
&= \ln\left[\lim_{s\to\infty}\frac{s-1}{s+1}\right] + \ln 3 = \ln 1 + \ln 3 = \ln 3
\end{aligned}$$

Here we have noticed the ln is continuous and so we can move the limit "inside." This shows that the given integral converges and it converges to $\ln 3$