Two Pages! **Quiz 5A**, Math 1860-021
2/27/14 Solutuions Name ____________________

(12) 1. Express the integrand as a sum of partial fractions and evaluate the integral. $\displaystyle\int_1^{\sqrt{3}} \frac{1}{t^3+t}\,dt$

The integrand $1/(t^3+t)$ already has the degree on bottom 3 higher than that on top (0) and moreover the bottom factors as $t^3+t = t(t^2+1)$. Therefore we can already apply the partial fractions expansion which is of the form

$$\frac{1}{t^3+t} = \frac{A}{t} + \frac{Bt+C}{t^2+1}$$

for some constants A, B and C. To solve for these constants we multiply by $t(t^2+1)$ and collect like coefficients.

$$1 = A(t^2+1) + (Bt+C)t = (A+B)t^2 + Ct + A$$

Equate like coefficients:
$$\begin{array}{rcl} A + B & = & 0 \\ C & = & 0 \\ A & = & 1 \end{array}$$

Clearly $A = 1$, $C = 0$ and $B = -1$. Therefore

$$\frac{1}{t^3+t} = \frac{1}{t} - \frac{t}{t^2+1}$$

Integrate each of the two terms separately

$$\int_1^{\sqrt{3}} \frac{1}{t^3+t}\,dt = \int_1^{\sqrt{3}} \frac{1}{t}\,dt - \int_1^{\sqrt{3}} \frac{t}{t^2+1}\,dt$$

In the second integral we substitute $u = t^2+1$ so that $du = 2t\,dt$ and the endpoints of integration become $u(1) = 2$ and $u(\sqrt{3}) = (\sqrt{3})^2 + 1 = 4$

$$\begin{aligned} \int_1^{\sqrt{3}} \frac{1}{t^3+t}\,dt &= \ln t|_1^{\sqrt{3}} - \frac{1}{2}\int_2^4 \frac{1}{u}\,du \\ &= \ln\sqrt{3} - \ln 1 - \frac{1}{2}\ln u|_2^4 = \frac{1}{2}\ln 3 - \frac{1}{2}\ln 4 + \frac{1}{2}\ln 2 = \frac{1}{2}[\ln 3 - \ln 2] \end{aligned}$$

(8) 2. Evaluate the integral. $\displaystyle\int_2^{\infty} \frac{2}{v^2-v}\,dv$

Again this integral is by partial fractions. Division is not required. The bottom factors as $v^2 - v = v(v-1)$ and so the partial fractions expansion is

$$\frac{2}{v^2-v} = \frac{2}{v(v-1)} = \frac{A}{v} + \frac{B}{v-1}$$

To find the actual constants A and B, multiply both sides by $v(v-1)$ and collect up like coefficients.

$$2 = A(v-1) + Bv = (A+B)v - A$$

Equating coefficients gives $A + B = 0$ and $-A = 2$ so that $A = -2$ and $B = 2$. Therefore

$$\frac{2}{v^2 - v} = -\frac{2}{v} + \frac{2}{v-1}$$

Integrate.

$$\begin{aligned}
\int_2^\infty \frac{2}{v^2 - v}\, dv &= \lim_{t\to\infty} \int_2^t -\frac{2}{v} + \frac{2}{v-1}\, dv \\
&= \lim_{t\to\infty} \left[-2\ln|v| + 2\ln|v-1|\right]_2^t \\
&= \lim_{t\to\infty} 2\ln\left|\frac{v-1}{v}\right|_2^t \\
&= \lim_{t\to\infty} 2\ln\left[\frac{t-1}{t}\right] - 2\ln\frac{1}{2} \\
&= 2\ln\left[\lim_{t\to\infty}\frac{t-1}{t}\right] + 2\ln 2 = 2\ln 1 + 2\ln 2 = 2\ln 2
\end{aligned}$$

Here we have noticed the ln is continuous and so we can move the limit “inside.” This shows that the given integral converges and it converges to $2\ln 2$