Final Exam, Math 1860

December, 2013 Solutions Name ________________

A non graphing calculator is permitted but no calculator is needed. A formula sheet is also allowed.

Common Assessment Questions: 3, 4, 6, 10 , 13 (Marked with *)

(31) 1. Evaluate the integral.

(a) $\int xe^{2x}\,dx$

Solution: Integrate by parts: $u = x$, $dv = e^{2x}\,dx$ so that $du = dx$ and $v = (1/2)e^{2x}$. Recall $\int u\,dvuv - \int v\,du$ so that

$$\int xe^{2x}\,dx \;=\; \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

Check by differentiation using the product rule:

$$\frac{d}{dx}\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} = \frac{1}{2}xe^{2x}2 + \frac{1}{2}e^{2x} - \frac{1}{4}(e^{2x})2 = xe^{2x}$$

which is the original integrand.

(b) $\int \sqrt{4-x^2}\,dx$

Solution: Trigonometric substitution: $x = 2\sin\theta$, so that $dx = 2\cos\theta$ and $\sqrt{4-x^2} = \sqrt{4-4(\sin\theta)^2} = 2\cos\theta$. See diagram.

$$\begin{aligned}
\int \sqrt{4-x^2}\,dx &= \int 2\cos\theta 2\cos\theta\,d\theta \\
&= 4\int \frac{1}{2}(1+\cos 2\theta)\,d\theta \\
&= 2[\theta + \frac{1}{2}\sin 2\theta] + C \\
&= 2[\sin^{-1}(x/2) + \sin\theta\cos\theta] + C \\
&= 2[\sin^{-1}(x/2) + (x/2)(\sqrt{4-x^2})/2 + C \\
&= 2\sin^{-1}(x/2) + \frac{1}{2}x\sqrt{4-x^2} + C
\end{aligned}$$

where we have used the identities $\cos^2\theta = (1/2)(1+\cos 2\theta)$ and $\sin 2\theta = 2\sin\theta\cos\theta$. The step of converting back into terms of x using a right triangle with angle θ and opposite side x and hypotenuse 2 and adjacent side $\sqrt{4-x^2}$ (diagram). Check by differentiation.

$$\frac{d}{dx}[2\sin^{-1}(x/2) + \frac{1}{2}x\sqrt{4-x^2}]$$

$$= \frac{2}{(1-x^2/4)^{1/2}}(1/2) + \frac{1}{2}x(1/2)(4-x^2)^{-1/2}(-2x) + \frac{1}{2}\sqrt{4-x^2}$$
$$= \frac{2}{\sqrt{4-x^2}} - \frac{1}{2}x^2(4-x^2)^{-1/2} + \frac{1}{2}\sqrt{4-x^2}$$
$$= \frac{1}{2}\frac{4-x^2}{\sqrt{4-x^2}} + \frac{1}{2}\sqrt{4-x^2} = \sqrt{4-x^2}$$

(c) $\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx$

Solution: Apply partial fractions. The bottom factors as $x^3+2x^2 = x^2(x+2)$. Therefore the integrand can be written as

$$\frac{5x^2+3x-2}{x^3+2x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

Multiply through by the divisor $x^2(x+2)$

$$5x^2+3x-2 = Ax(x+2) + B(x+2) + Cx^2 = (A+C)x^2 + (2A+B)x + 2B$$

Equate coefficients. $\begin{array}{ccccccc} A & & & + & C & = & 5 \\ 2A & + & B & & & = & 3 \\ & & 2B & & & = & -2 \end{array}$

so that $B=-1$ and so $A=2$ by the second equation and $C=3$:

$$\frac{5x^2+3x-2}{x^3+2x^2} = \frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}$$

This can be checked by finding a common denominator. Integrate.

$$\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx \int \frac{2}{x}\,dx - \int x^{-2}\,dx + 3\int \frac{1}{x+2}\,dx = 2\ln|x| + x^{-1} + 3\ln|x+2| + C$$

and this checks by differentiation.

$$\frac{d}{dx} 2\ln|x| + x^{-1} + 3\ln|x+2| = \frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}$$

2. Set up, but do NOT evaluate an integral that represents the volume of the solid obtained by rotating the region bounded by the curve $y = \ln x$, and two lines, $y = 0$ and $x = 3$
(10) about the y-axis. Suggestion: Sketch the region.

Solution: Graph the logarithm curve and the vertical line $x=3$ and the x-axis ($y=0$) enclose a triangular region. The curve $y=\ln x$ crosses the x-axis when $0=\ln x$ so that $x=1$. The volume obtained by rotating about the y-axis is most easily described by cylindrical shells:

$$2\pi \int_1^3 x\ln x\,dx$$

It is possible to find the volume using the "washer method" too.

$$\int_0^1 \pi e^2 - \pi(e^y)^2\,dy = \pi[e^2 - \int_0^1 e^{2y}\,dy] = \dots$$

(10) 3. Determine whether the integral is convergent or divergent. If it is convergent then evaluate it.

$$\int_1^\infty \frac{2x}{1+x^4}\,dx$$

Solution: Write the improper integral as a limit and use u-substitution. Let $u = x^2$ so taht $du = 2x\,dx$

$$\begin{aligned}
\int_1^\infty \frac{2x}{1+x^4}\,dx &= \lim_{t\to\infty} \int_1^t \frac{2x}{1+x^4}\,dx \\
&= \lim_{t\to\infty} \int_{x=1}^{x=t} \frac{1}{1+u^2}\,du \\
&= \lim_{t\to\infty} \tan^{-1} u|_{x=1}^{x=t} \\
&= \lim_{t\to\infty} \tan^{-1} x^2|_1^t \\
&= \lim_{t\to\infty} \tan^{-1} t^2 - \tan^{-1} 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}
\end{aligned}$$

Therefore the improper integral converges to $\pi/4$.

(10) 4. * Find an equation for the tangent line to the curve $x = te^t$, $y = t + e^t$, at the point corresponding to $t = 0$.

Solution: When $t = 0$, $x = 0$ and $y = 1$ so that we want a line through (0,1) and slope

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1+e^t}{te^t + e^t}$$

When $t = 0$ we have the slope is $dy/dx = 2$ and so an equation for the tangent line is

$$y - 1 = 2(x - 0) \quad \text{or} \quad y = 2x + 1$$

(10) 5. Find the length of the curve $x = e^t \cos t$ and $y = e^t \sin t$, $0 \le t \le 2$.

Compute $x' = e^t(\cos t - \sin t)$, $y' = e^t(\sin t + \cos t)$ by the product rule. Therefore

$$\begin{aligned}
((x')^2 + (y')^2)^{1/2} &= \left(e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2\right)^{1/2} \\
&= e^t \left((\cos t)^2 - 2\cos t \sin t + (\sin t)^2 + (\sin t)^2 + 2\sin t \cos t + (\cos t)^2\right)^{1/2} \\
&= \sqrt{2}e^t
\end{aligned}$$

so that the length of the curves is

$$\int_0^2 ((x')^2 + (y')^2)^{1/2}\,dt = \int_0^2 \sqrt{2}e^t\,dt = \sqrt{2}e^t|_0^2 = \sqrt{2}(e^2 - 1)$$

(12) 6. Determine whether the series converges or diverges. If it converges then find the sum.

$$7 - \frac{14}{3} + \frac{28}{9} - \frac{56}{27} + \frac{112}{81} - \frac{224}{243} + \dots$$

Solution: This is a geometric series with $a = 7$ and $r = -2/3$. Since $|r| = 2/3 < 1$ the series converges to $a/(1-r) = 7/(1-(-2/3)) = 21/5$.

(16) 7. (a) Sketch the curve with polar equation $r = \cos 2\theta$.

Solution: Graph $y = \cos 2x$ in Cartesian coordinates or tabulate values.

(b) Set up, but do NOT evaluate an integral that represents the area enclosed by one loop of the curve $r = \cos 2\theta$ of part (a).

Solution: From part (a) it is clear that one loop is traced out if $-\pi/4 < \theta < \pi/4$ and so the area is

$$\frac{1}{2}\int_{-\pi/4}^{\pi/4} (\cos 2\theta)^2 \, d\theta$$

(16) 8. * Determine whether the series is convergent or divergent. Explain your reasoning.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^3+5}}$

Solution: Compare this to the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^{3/2}}$. This latter series is a p-series with $p = 3/2$. Since $p > 1$ this latter series converges. Since

$$\frac{2}{\sqrt{n^3+5}} \leq \frac{2}{\sqrt{n^3}} = \frac{2}{n^{3/2}}$$

that is our series is smaller than the convergent series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^{3/2}}$, our series must also be convergent by the comparision test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n(n!)}$

Solution: Apply the ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{5^{n+1}}{(n+1)(n+1)!}}{\frac{5^n}{n(n!)}}\right| = \lim_{n\to\infty}\frac{5^{n+1}n(n!)}{5^n(n+1)(n+1)!} = \lim_{n\to\infty}\frac{5n}{(n+1)^2} = 0$$

and since $0 < 1$, the series converges by the ratio test.

(18) 9. Determine whether the series is absolutely convergent, conditionally convergent or divergent. Explain your reasoning.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n}$

Solution: This is an alternating series and since $1/(3n)$ decreases to 0, this series converges by the alternating series test. Therefore the series converges conditionally at least. Consider the corresponding series with absolute values

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{3n} \right| = \sum_{n=1}^{\infty} \frac{1}{3n}$$

then we see that this series is the harmonic series (p-series with $p = 1$) (times $1/3$) and is therefore divergent. Therfore the original series is only condtionally convergent and not absolutely convergent.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{n+3}$

Solution This too is an alteranting series but this time the terms do not converge to 0 because

$$\lim_{n\to\infty} \frac{2n+1}{n+3} = \lim_{n\to\infty} \frac{n}{n} \frac{2+1/n}{1+3/n} = \lim_{n\to\infty} \frac{2+1/n}{1+3/n} = 2$$

Therefore the series diverges.

(12) 10. Find the interval of convergence of the series. (You need not check convergence at the endpoints.)

$$\sum_{n=2}^{\infty} \frac{n-1}{2^n}(x-1)^n$$

Solution: Apply the ratio test.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{n}{2^{n+1}}(x-1)^{n+1}}{\frac{n-1}{2^n}(x-1)^n} \right| = \lim_{n\to\infty} \frac{n}{n-1}\frac{1}{2}|x-1| = \frac{1}{2}|x-1|$$

Therefore the series converges absolutely provided $\frac{1}{2}|x-1| < 1$ by the ratio test, that is provided $|x-1| < 2$ so that the interval of convergence is $-1 < x < 3$.

(12) 11. * Find the Taylor polynomial $T_n(x)$ of $f(x) = \sin x$ at $a = \pi/2$ (that is in powers of $x - \pi/2$) of degree $n = 4$.

Solution: We need to calculate the first few derivatives of $f(x)$ and evaluate them at $\pi/2$.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\sin x$	1
1	$\cos x$	0
2	$-\sin x$	-1
3	$-\cos x$	0
4	$\sin x$	1

and therefore the Taylor polynomial is

$$T_4(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = 1 - \frac{1}{2!}(x-\pi/2)^2 + \frac{1}{4!}(x-\pi/2)^4$$

12. Recall that the Maclaurin series expansion for $\cos x$ is

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

(12) Evaluate the indefinite integral below as an infinite power series (in powers of x.)

$$\int \cos(3x^2)\,dx$$

Solution: First we observe that we can substitute $3x^2$ into the given series expansion for $\cos x$ and get the power series expansion for $\cos x$

$$\cos 3x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{4n}}{(2n)!} = 1 - \frac{9x^4}{2!} + \frac{3^4x^8}{4!} - \frac{3^6x^{12}}{6!} + \dots$$

Then we can integrate term by term.

$$\int \cos(x^2)\,dx = C + x - \frac{9x^5}{5\cdot 2!} + \frac{3^4x^9}{9\cdot 4!} - \frac{3^6x^{13}}{13\cdot 6!} + \dots$$

(6) 13. Describe in words the region of $\mathbb{R}^3$ represented by $x^2+y^2+z^2 < 4z$.

Complete the square. The inequality is $x^2+y^2+z^2 < 4z$ or $x^2+y^2+z^2-4z+(-2)^2 < (-2)^2$ that is $x^2+y^2+(z-2)^2 < 4$ and this inequality says that the square of the distance of (x,y,z) to (0,0,2) is less than 4. Therefore the region is the interior of a ball of radius 2 and center (0,0,2).

(7) 14. Find the scalar and vector projection of $\vec{b} = \vec{i} + \vec{j} + \vec{k}$ onto $\vec{a} = -\vec{i} + \vec{j} - \vec{k}$.

The scalar projection of $\vec{b}$ onto $\vec{a}$ is

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{-1}{\sqrt{3}}$$

and the vector projection is

$$\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a} = \frac{-1}{3}(-\vec{i}+\vec{j}-\vec{k})$$

15. (a) Find a unit vector orthogonal to the plane through $P(0,1,0)$, $Q(2,2,1)$ and $R(3,1,-1)$,
(8) and (b) find the area of triangle PQR.

Compute the vectors that make up two of the edges of the triangle. $\vec{PQ} = \langle 2,1,1\rangle$ and $\vec{PR} = \langle 3,0,-1\rangle$. (Either edge could be replaced by $\vec{QR}$.) To get a vector orthogonal to the plane we take the cross product.

$$\begin{aligned} \overrightarrow{PQ}\times\overrightarrow{PR} &= \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 3 & 0 & -1 \end{bmatrix} = (-1-0)\vec{i} - (-2-3)\vec{j} + (0-3)\vec{k} \\ &= -\vec{i} + 5\vec{j} - 3\vec{k} \end{aligned}$$

and the length of this vector is $|-\vec{i}+5\vec{j}-3\vec{k}|=\sqrt{35}$.

(a)Therefore the two unit vectors orthogonal to the plane are

$$\pm\frac{1}{\sqrt{35}}(-\vec{i}+5\vec{j}-3\vec{k})$$

(b) and the area of the triangle PQR is $\sqrt{35}/2$.

16. Find an equation for the plane through (2,-3,4) that contains the line $x=4-2t$,
(8) $y=1+t$ and $z=3t$.

As in the previous question we need two vectors in the plane. One is the direction vector $\langle -2,1,3\rangle$ of the line. Another is the vector $\langle 2,4,-4\rangle$ from (2,-3,4) to a point on the line (4,1,0) (take $t=0$). The cross product is

$$\begin{aligned}\overrightarrow{PQ}\times\overrightarrow{PR} &= \det\begin{bmatrix}\vec{i} & \vec{j} & \vec{k}\\ -2 & 1 & 3\\ 2 & 4 & -4\end{bmatrix}=(-4-12))\vec{i}-(8-6)\vec{j}+(-8-2)\vec{k}\\ &= -16\vec{i}-2\vec{j}-10\vec{k}\end{aligned}$$

and this vector is a normal to the plane. An equation for the plane is $-16(x-2)-2(y+3)-10(z-4)=0$ or $8x+y+5z=33$.