 Review Test 2, Math 1850 Section 3 or 4
11/2/11 SOLUTIONS Name ______________

(6 each) 1. Find the derivative of y with respect to the given variable.

(a) $y = \sqrt{x}e^{2x}$
We write $y = x^{1/2}e^{2x}$ and differentiate using the product rule.

$$\begin{aligned} y' &= x^{1/2}\frac{d}{dx}e^{2x} + e^{2x}\frac{d}{dx}x^{1/2} \\ &= x^{1/2}e^{2x}2 + e^{2x}\frac{1}{2}x^{-1/2} = e^{2x}\left[2x^{1/2} + \frac{1}{2x^{1/2}}\right] = \frac{e^{2x}}{2x^{1/2}}[4x+1] \end{aligned}$$

(b) $y = \dfrac{\ln x}{3 + \ln x}$
The quotient rule applies here.

$$\begin{aligned} y' &= \frac{(3+\ln x)\frac{d}{dx}\ln x - \ln x\frac{d}{dx}(3+\ln x)}{(3+\ln x)^2} \\ &= \frac{(3+\ln x)(1/x) - \ln x(1/x)}{(3+\ln x)^2} = \frac{3}{x(3+\ln x)^2} \end{aligned}$$

(c) $y = \tan^{-1}(x/3) - 5\sin^{-1}(x/2)$
By the chain rule

$$y' = \frac{1}{1+(x/3)^2}\,\frac{1}{3} - 5\frac{1}{\sqrt{1-(x/2)^2}}\,\frac{1}{2} = \frac{3}{9+x^2} - \frac{5}{\sqrt{4-x^2}}$$

(d) $y = 5\log_3 x + 10^{\sqrt{x}}$
Alternatively $y = 5\log_3 x + 10^{x^{1/2}}$. The chain rule applies to the second term.

$$y' = \frac{5}{x\ln 3} + (\ln 10)10^{x^{1/2}}(1/2)x^{-1/2} = \frac{5}{x\ln 3} + \frac{\ln 10}{2x^{1/2}}10^{x^{1/2}}$$

(9) 2. Use implicit differentiation to find dy/dx if

$$\frac{1}{x} + x\cos y + \sin 2y = 0$$

The given equation determines y as a function of x. We differentiate in x: the first term is x^{-1}; in the second term we apply the product rule and in the third term the chain rule

$$\begin{aligned} -x^{-2} + x\frac{d}{dx}\cos y + \cos y\frac{d}{dx}x + \cos 2y\frac{d2y}{dx} &= 0 \\ -x^{-2} + x(-\sin y)\frac{dy}{dx} + \cos y + 2\cos 2y\frac{dy}{dx} &= 0 \end{aligned}$$

$$\begin{aligned}
[-x\sin y + 2\cos 2y]\frac{dy}{dx} &= x^{-2} - \cos y \\
\frac{dy}{dx} &= \frac{x^{-2} - \cos y}{-x\sin y + 2\cos 2y} \\
\frac{dy}{dx} &= \frac{1 - x^2\cos y}{x^2(2\cos 2y - x\sin y)}
\end{aligned}$$

(8) 3. Find the differential of the function $y = (x^3+4)^{1/3}$

Solution: By the generalized power rule, the derivative is

$$\frac{dy}{dx} = \frac{1}{3}(x^3+4)^{-2/3}3x^2 = x^2(x^3+4)^{-2/3}$$

so that the differential is

$$dy = x^2(x^3+4)^{-2/3}\,dx$$

(8) 4. Find the linearization $L(x)$ of $f(x) = \sec(2x)$ at $x = \pi/8$.

The linearization of f at a is $L(x) = f(a) + f'(a)(x-a)$. Here $f'(x) = \sec(2x)\tan(2x)2 = 2\sec(2x)\tan(2x)$ and $a = \pi/8$ so that $f(a) = f(\pi/8) = \sec(\pi/4) = \sqrt{2}$ and $f'(a) = f'(\pi/8) = 2\sec(\pi/4)\tan(\pi/4) = 2\sqrt{2}$ so that

$$L(x) = \sqrt{2} + 2\sqrt{2}(x - \pi/8)$$

(8) 5. Find the derivative of $y = x^{\sin x}$, where $x > 0$. (Suggestion: Use logarithmic differentiation.)

Take ln of both sides: $\ln y = \ln x^{\sin x} = \sin x \ln x$. Differentiate: by the product rule

$$\frac{y'}{y} = \sin x(1/x) + \ln x(\cos x)$$

and solving for y' we have

$$y' = \left(\frac{\sin x}{x} + \cos x \ln x\right) y = \left(\frac{\sin x}{x} + \cos x \ln x\right) x^{\sin x}$$

(11) 6. Find the absolute maximum and minimum of $f(x) = -2x^3 + 6x^2 - 3$ on the interval [-1,4]. Show your reasoning.

Solution Here f is continuous on a closed interval [-1,4] so that we can use the closed interval method. We will need $f'(x) = -6x^2 + 12x$.

(a) Check for critical points. First set $f'(x) = 0$ so that $-6x^2 + 12x = -6x(x-2) = 0$. Therefore $x = 2$ and $x = 0$ are two critical points. Since $f'(x)$ is defined everywhere there are no other critical points.

(b) The endpoints are $x = -1$ and $x = 4$.

(c) Evaluate.

Critical and End Points a	Evaluate $f(a)$	Absolute Extrema?
$a=-1$	$f(-1)=5$	Abs Max
$a=0$	$f(0)=-3$	
$a=2$	$f(2)=5$	
$a=4$	$f(4)=-35$	Abs Min

so that the absolute maximum value is 5 and it occurs when $x=-1$ and $x=2$ and the absolute minimum is -35 and it occurs at $x=4$.

7. Graph the function below. Determine the local maxima, minima and inflection points as well as intervals of increase and decrease.

(18)

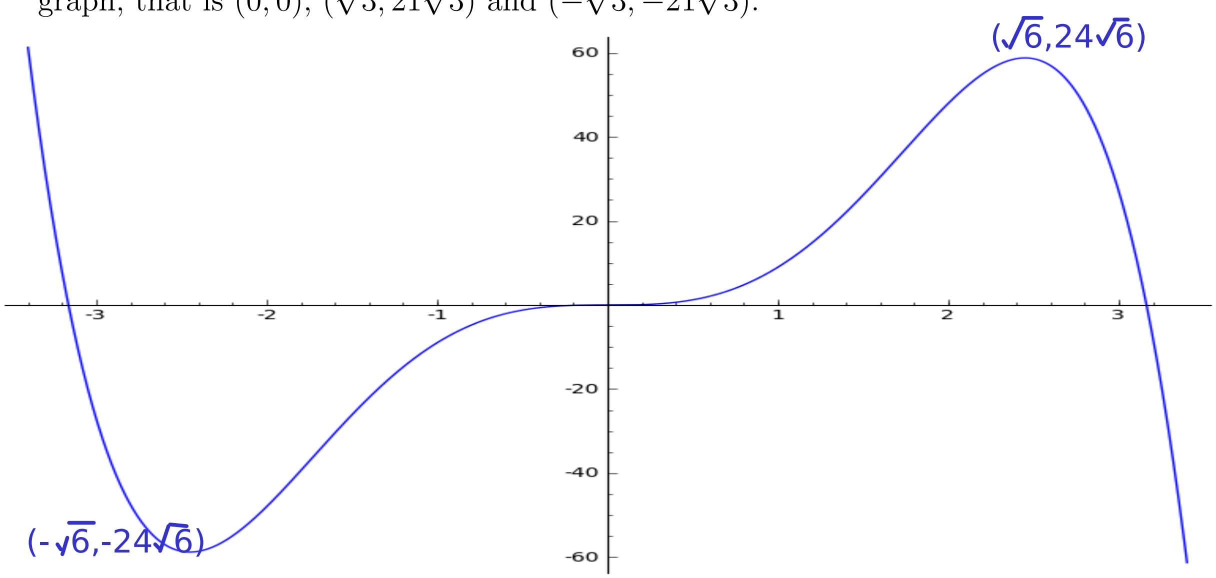
$$y = 10x^3 - x^5$$

Solution By Fermat's theorem, local extrema occur at critical points. Differentiate: $f'(x) = 30x^2 - 5x^4 = 5x^2(6-x^2)$. Certainly f' is defined everywhere and so the critical points are where $f'(t)=0$ and $5t^2(6-t^2)=0$ implies $t=0$ or $t=\pm\sqrt{6}$. We check now for intervals of increase and decrease by the first derivative test.

Interval	Evaluate f'	Increasing or Decreasing
$t<-\sqrt{6}$	$f'(-3)<0$	Decreasing
$-\sqrt{6}<t<0$	$f'(-1)>0$	Increasing
$0<x<\sqrt{6}$	$f'(1)>0$	Increasing
$\sqrt{6}<t$	$f'(3)<0$	Decreasing

Therefore $t=-\sqrt{6}$ is a local min with local minimum value $f(-\sqrt{6})=-24\sqrt{6}$ and $t=\sqrt{6}$ is a local max with local max value $f(\sqrt{6})=24\sqrt{6}$ and $t=0$ is neither a local max nor min.

We also check for any inflection points. Since $f''(x)=60x-20x^3=20x(3-x^2)$ The inflection points occur when $f''(x)=0$: $20x(3-x^2)=0$ so that $x=0$ or $x=\pm\sqrt{3}$ The inflection points are the corresponding points $(x,f(x))$ on the graph, that is $(0,0)$, $(\sqrt{3},21\sqrt{3})$ and $(-\sqrt{3},-21\sqrt{3})$.



8. A paper cup has the shape of a cone with height 12 cm and radius 4 cm (at the top). If water is poured into the cup at 2 cm^3/sec, how fast is the water level rising when the water is 8 cm deep? (Recall that the volume of a cone of height h and radius r is $\pi r^2 h/3$.)

(14) Draw a picture. Let $V = \pi r^2 h/3$ be the volume of the water in the cup. We know $dV/dt = 2$. Let h be the depth of the water. We want dh/dt. But what is r? By similar triangles $r/4 = h/12$ (the ratio of the radius of the water to the radius of the cup is equal to the ratio of the depth of the water to the height of the cup.) Therefore

$$V = \frac{\pi}{3}\left(\frac{4h}{12}\right)^2 h = \frac{\pi}{27}h^3$$

Differentiate in t.

$$V' = \frac{\pi}{27}3h^2h' = \frac{\pi}{9}h^2h'$$

Evaluate when $h = 8$ we have

$$2 = \frac{\pi}{9}8^2h' \quad \text{or } h' = \frac{9}{32\pi}$$

The level of water in the cup is rising at $9/(32\pi)$ cm/sec.