 Test 2B, Math 1850 Section 012
11-6-13 Solutions Name ______________________

1. Find the derivative of the functions. (21)

   (a) $y = e^{2x}\cos 3x$
   By the product rule (and chain rule)
   $y' = e^{2x}(-\sin 3x)(3) + \cos 3x e^{2x}(2) = -3e^{2x}\sin 3x + 2\cos 3x e^{2x} = e^{2x}(-3\sin 3x + 2\cos 3x)$

   (b) $g(s) = 10^s + \log_3 s$
   We have
   $$g'(s) = (\ln 10)10^s + \frac{1}{(\ln 3)s}$$

   (c) $f(t) = \sin^{-1}(t^3) + \tan^{-1}(lnt)$
   Here the chain rule applies
   $$f'(t) = \frac{1}{\sqrt{1-(t^3)^2}}\,3t^2 + \frac{1}{1+(\ln t)^2}\,\frac{1}{t} = \frac{3t^2}{\sqrt{1-t^6}} + \frac{1}{t(1+(\ln t)^2)}$$

2. Find the differential of the function $y = \sec(x^2)$. (8)
   Differentiate $y = \sec(x^2)$ using the chain rule.
   $$y' = \sec(x^2)\tan(x^2)2x = 2x\sec(x^2)\tan(x^2)$$
   In differential notation
   $$dy = 2x\sec(x^2)\tan(x^2)\,dx$$

3. Use implicit differentiation to find dy/dx if $e^y - x^3y = 1$. (8)
   The given equation determines y as a function of x. Differentiate in x
   $$e^y\frac{dy}{dx} - 3x^2y - x^3\frac{dy}{dx} = 0$$
   where the chain rule was applied to the first term and the product rule to the second term. Now solve for dy/dx.
   $$(e^y - x^3)\frac{dy}{dx} = 3x^2y, \quad \text{or} \quad \frac{dy}{dx} = \frac{3x^2y}{e^y - x^3}$$

4. Find the linearization $L(x)$ of $f(x) = 2x/(x+1)$ at $a = 1$. (8)
   Recall $L(x) = f(a) + f'(a)(x-a)$ (that is the formula for the tangent line). Here, by the quotient rule
   $$f'(x) = \frac{2(x+1) - 2x}{(x+1)^2} = \frac{2}{(x+1)^2}$$
   Evaluate $f(a) = f(1) = 2/2 = 1$ and $f'(1) = 2/4 = 1/2$ so that
   $$L(x) = 1 + \frac{1}{2}(x-1)$$

5. Use logarithmic differentiation to find dy/dx. (9)

$$y = x^{\sqrt{x}}$$

Take the natural logarithm of both sides and simplify

$$\ln y = \ln\left(x^{\sqrt{x}}\right) = x^{1/2}\ln x$$

Now we differentiate both sides.

$$\frac{y'}{y} = x^{1/2}(1/x) + (1/2)x^{-1/2}\ln x = \frac{2+\ln x}{2x^{1/2}}$$

by the product rule and then finding a common denominator. Now we solve for y' by multiplying both sides by y and then substituting for y.

$$y' = \left(\frac{2-\ln x}{2x^{1/2}}\right)y = \left(\frac{2+\ln x}{2x^{1/2}}\right)x^{\sqrt{x}}$$

6. Find the *absolute* maximum and minimum of $f(x) = x^3 - 6x - 2$ on the closed
(12) and bounded interval $-2 \le x \le 2$.

This uses the "closed interval method." We differentiate: $f'(x) = 3x^2 - 6 = 3(x^2-2)$ Step One is to find the critical points. We set $f'(x) = 3(x-\sqrt{2})(x+\sqrt{2})$ to equal 0 and find $x = \pm\sqrt{2}$. These are the only critical points because $f'(x)$ is defined for all x. Also both critical points $x = \pm\sqrt{2}$ lie in the given interval $-2 \le x \le 2$. Step Two is to identify the endpoints: $x = \pm 2$.

Step Three is to evaluate at all the points:

Critical and End Points	Evaluate $f(x)$	Absolute Max/Min?
$x = -2$	$f(-2) = 2$	
$x = -\sqrt{2}$	$f(-\sqrt{2}) = 4\sqrt{2} - 2 \approx 3.66$	Abs. Max
$x = \sqrt{2}$	$f(\sqrt{2}) = -4\sqrt{2} - 2$	Abs. min
$x = 2$	$f(2) = -6$	

So that $x = -\sqrt{2}$ is the absolute maximum point with value $4\sqrt{2}-2$ and $x = \sqrt{2}$ is the absolute minimum point with value $-4\sqrt{2} - 2$

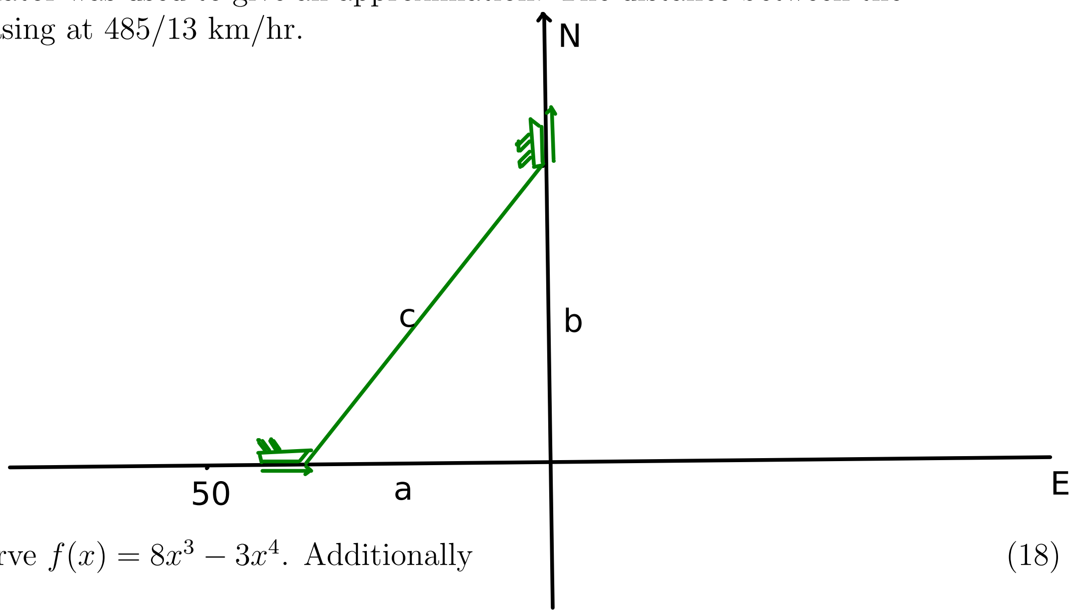
7. At noon, ship A is 50 km west of ship B. Ship A is sailing east at 25 km/hr and ship B is sailing north at 30 km/hr. How fast is the distance between the ships
(16) changing at 4:00 P.M?

A sketch might place ship B at the origin at noon. It cruises north up the y-axis at 30 km/hr. Ship A starts at 50 km west of the origin: that is - 50 on the x-axis. It travels at 25 km/hr east which is in the positive direction on the x-axis. Let $a = a(t)$ be the position of ship A on the x-axis and b of ship B: we know $a' = 25$ and $b' = 30$. We are asked for the rate at which the distance between the ships is changing and so we let c be the distance between the ships and we want $c'(4)$. (Time is in hours after noon.)

Relate a, b and c: $c^2 = a^2 + b^2$, by the Pythagorean theorem. Differentiate in t: $2cc' = 2aa' + 2bb'$ or $cc' - aa' + bb'$. Consider 4 P.M.: $t = 4$. We want $c'(4)$. We know $a'(4) = 25$, $b'(4) = 30$ but what is $a(4)$? Ship A has traveled 4(25) km and so $a(4) = 50$. Similary $b(4) = 30(4) = 120$. and so $c(4) = \sqrt{a(4)^2 + b(4)^2} = \sqrt{50^2 + 120^2} = 130$. Therefore the distance between ships is increasing

$$c'(4) = \frac{aa' + bb'}{c} = \frac{(50)(25) + 120(30)}{130} = \frac{485}{13} \approx 37.3$$

where a calculator was used to give an approximation. The distance between the ships is increasing at 485/13 km/hr.



8. Sketch the curve $f(x) = 8x^3 - 3x^4$. Additionally (18)

   (a) Find intervals of increase and decrease.

   (b) Find the local maxima and minima.

   (c) Find the intervals of concavity

We will need the first two derivatives: $f'(x) = 24x^2 - 12x^3 = 12x^2(2 - x)$ and $f''(x) = 48x - 36x^2 = 12x(4 - 3x)$. The critical points x occur when $f'(x) = 0$ or $f'(x)$ does not exist. But $12x^2(2 - x) = 0$ implies $x = 0$ or $x = 2$ and these are the only critical points. The intervals of increase and decrease are

Interval	Evaluate f'	Increasing or Decreasing
$x < 0$	$f'(-1) = 36 > 0$	Incr
$0 < x < 2$	$f'(1) = 12 > 0$	Incr
$2 < x$	$f'(3) = -108 < 0$	Decr

It follows that $x = 2$ is a local maximum where $f(2) = 8(2^3) - 3(2^4) = 16$. Check next concavity. The endpoints of intervals of concavity is where $f''(x) = 0$ (inflection point) or $f''(x)$ does not exist (or an endpoint of the domain). Here we set $f''(x) = 0$ to find $12x(4 - 3x) = 0$ so that $x = 0$ or $x = 4/3$ corresponding to inflection points (0,0) and (4/3, 256/27). ($f(-4/3) = -256/27$.) The intervals of concavity are

Interval	Evaluate f''	Concave Up or Down
$x < -4/3$	$f''(-2) = 46 < 0$	Up
$-4/3 < x < 0$	$f''(-1) = -12 < 0$	Down
$0 < x$	$f''(1) = 84 > 0$	Up

Now sketch the graph.