 Test 2A, Math 1850 Section 011
11-6-14 Solutions Name ____________

1. Find the derivative of the functions. (21)

   (a) $y = e^{3x}\sin 5x$
   By the product rule (and chain rule)

   $$y' = e^{3x}\cos 5x(5) + \sin 5x e^{3x}(3) = 5e^{3x}\cos 5x + 3\sin 5x e^{3x} = e^{3x}(5\cos 5x + 3\sin 5x)$$

   (b) $g(s) = 3^s + \log_{10} s$
   We have

   $$g'(s) = (\ln 3)3^s + \frac{1}{(\ln 10)s}$$

   (c) $f(t) = \sin^{-1}(\ln t) + \tan^{-1}(t^2)$
   Here the chain rule applies

   $$f'(t) = \frac{1}{\sqrt{1-(\ln t)^2}}\,\frac{1}{t} + \frac{1}{1+(t^2)^2}\,2t = \frac{1}{t\sqrt{1-(\ln t)^2}} + \frac{2t}{1+t^4}$$

2. Find the differential of the function $y = \cot(\sqrt{x})$. (8)

   Differentiate $y = \cot(x^{1/2})$ using the chain rule.

   $$y' = -(\csc(x^{1/2}))^2\frac{1}{2}x^{-1/2} = -\frac{1}{2x^{1/2}\sin(x^{1/2}))^2}$$

   In differential notation

   $$dy = -\frac{dx}{2x^{1/2}\sin(x^{1/2}))^2}$$

3. Use implicit differentiation to find dy/dx if $e^y - x^2y = 1$. (8)

   The given equation determines y as a function of x. Differentiate in x

   $$e^y\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} = 0$$

   where the chain rule was applied to the first term and the product rule to the first term. Now solve for dy/dx.

   $$(e^y - x^2)\frac{dy}{dx} = 2xy, \quad \text{or} \quad \frac{dy}{dx} = \frac{2xy}{e^y - x^2}$$

4. Find the linearization $L(x)$ of $f(x) = 6x/(x+2)$ at $a = 1$. (8)

   Recall $L(x) = f(a) + f'(a)(x-a)$ (that is the formula for the tangent line). Here, by the quotient rule

   $$f'(x) = \frac{6(x+2) - 6x}{(x+2)^2} = \frac{12}{(x+2)^2}$$

   Evaluate $f(a) = f(1) = 6/3 = 2$ and $f'(1) = 12/9 = 4/3$ so that

   $$L(x) = 2 + \frac{4}{3}(x-1)$$

5. Find dy/dx. (Suggestion: Use logarithmic differentiation.) (9)

$$y = x^{1/x}$$

Take the natural logarithm of both sides and simplify

$$\ln y = \ln\left(x^{1/x}\right) = \frac{\ln x}{x}$$

Now we differentiate both sides.

$$\frac{y'}{y} = \frac{x(1/x) - (1)\ln x}{x^2} = \frac{1-\ln x}{x^2}$$

and solve for y' by multiplying both sides by y and then substituting for y.

$$y' = \left(\frac{1-\ln x}{x^2}\right) y = \left(\frac{1-\ln x}{x^2}\right) x^{1/x}$$

6. Find the *absolute* maximum and minimum of $f(x) = x^3 - 6x - 1$ on the closed and
(12) bounded interval $-2 \le x \le 2$.

This uses the "closed interval method." We differentiate: $f'(x) = 3x^2 - 6 = 3(x^2 - 2)$ Step One is to find the critical points. We set $f'(x) = 3(x-\sqrt{2})(x+\sqrt{2})$ to equal 0 and find $x = \pm\sqrt{2}$. These are the only critical points because $f'(x)$ is defined for all x. Also both critical points $x = \pm\sqrt{2}$ lie in the given interval $-2 \le x \le 2$. Step Two is to identify the endpoints: $x = \pm 2$.

Step Three is to evaluate at all the points:

Critical and End Points	Evaluate $f(x)$	Absolute Max/Min?
$x = -2$	$f(-2) = 3$	
$x = -\sqrt{2}$	$f(-\sqrt{2}) = 4\sqrt{2} - 1 \approx 4.66$	Abs. Max
$x = \sqrt{2}$	$f(\sqrt{2}) = -4\sqrt{2} - 1$	Abs. min
$x = 2$	$f(2) = -5$	

So that $x = -\sqrt{2}$ is the absolute maximum point with value $4\sqrt{2} - 1$ and $x = \sqrt{2}$ is the absolute minimum point with value $-4\sqrt{2} - 1$.

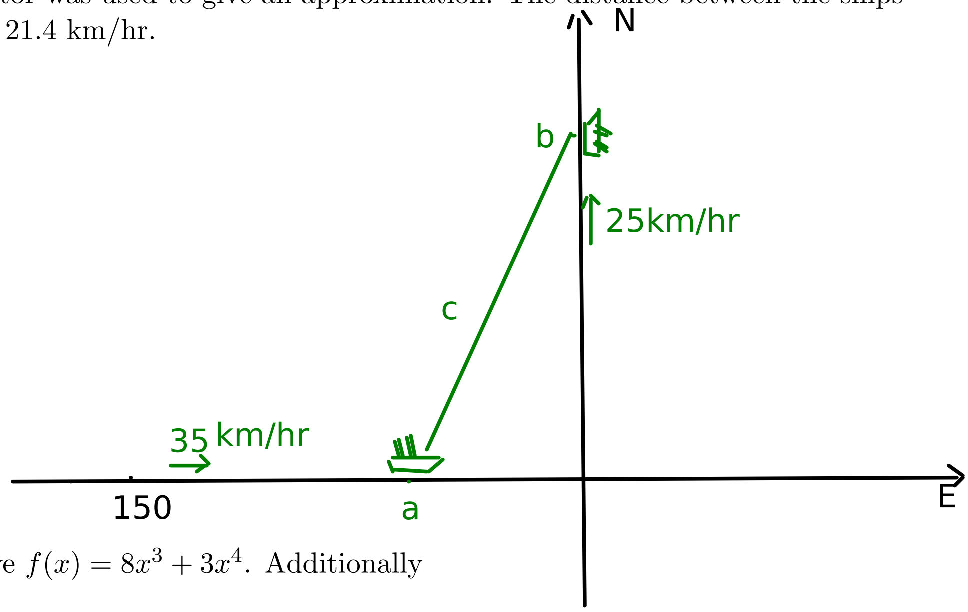
7. At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/hr and ship B is sailing north at 25 km/hr. How fast is the distance between the ships changing at
(16) 4:00 P.M?

A sketch might place ship B at the origin at noon. It cruises north up the y-axis at 25 km/hr. Ship A starts at 150 km west of the origin: that is -150 on the x-axis. It travels at 35 km/hr east which is in the positive direction on the x-axis. Let $a = a(t)$ be the position of ship A on the x-axis and b of ship B: we know $a' = 35$ and $b' = 25$. We are asked for the rate at which the distance between the ships is changing and so we let c be the distance between the ships and we want $c'(4)$. (Time is in hours after noon.)

Relate a, b and c: $c^2 = a^2 + b^2$, by the Pythagorean theorem. Differentiate in t: $2cc' = 2aa' + 2bb'$ or $cc' = aa' + bb'$. Consider 4 P.M.: $t = 4$. We want $c'(4)$. We know $a'(4) = 35$, $b'(4) = 25$ but what is $a(4)$? Ship A has traveled $4(35) = 140$ km from -150 $a(4) = -10$. Similarly $b(4) = 25(4) = 100$. and so $c(4) = \sqrt{a(4)^2 + b(4)^2} = \sqrt{10^2 + 100^2} \approx= 100.5$. Therefore the distance between ships is increasing

$$c'(4) = \frac{aa' + bb'}{c} = \frac{(-10)(35) + 100(25)}{100.5} \approx 21.4$$

where a calculator was used to give an approximation. The distance between the ships is increasing at 21.4 km/hr.



8. Sketch the curve $f(x) = 8x^3 + 3x^4$. Additionally (18)

   (a) Find intervals of increase and decrease.

   (b) Find the local maxima and minima.

   (c) Find the intervals of concavity

We will need the first two derivatives:$f'(x) = 24x^2 + 12x^3 = 12x^2(2 + x)$ and $f''(x) = 48x + 36x^2 = 12x(4 + 3x)$. The critical points x occur when $f'(x) = 0$ or $f'(x)$ does not exist. But $12x^2(2 + x) = 0$ implies $x = 0$ or $x = -2$ and these are the only critical points. The intervals of increase and decrease are

Interval	Evaluate f'	Increasing or Decreasing
$x < -2$	$f'(-3) = -108 < 0$	Decr
$-2 < x < 0$	$f'(-1) = 12 > 0$	Incr
$0 < x$	$f'(1) = 36 < 0$	Incr

It follows that $x = -2$ is a local minimum where $f(-2) = 8(-2^3) - 3(2^4) = -16$. Check next concavity. The endpoints of intervals of concavity is where $f''(x) = 0$ (inflection point) or $f''(x)$ does not exist (or an endpoint of the domain). Here we set $f''(x) = 0$ to find $12x(4 + 3x) = 0$ so that $x = 0$ or $x = -4/3$ corresponding to inflection points (0,0) and (-4/3,-256/27). ($f(-4/3) = -256/27$.) The intervals of concavity are

Interval	Evaluate f''	Concave Up or Down
$x < -4/3$	$f''(-2) = 46 < 0$	Up
$-4/3 < x < 0$	$f''(-1) = -12 < 0$	Down
$0 < x$	$f''(1) = 84 > 0$	Up

Now sketch the graph.