 Review Test 1, Math 1850 Section 7/8

Solutions Name ____________

(6) 1. Simplify the expressions

(a) $\log_3 \sqrt{3} = \log_3 3^{1/2} = \dfrac{1}{2}$

(b) $\ln(e^{\sin x}) = \sin x$

(3) 2. Solve for k in $e^{5k} = 1/32$.

Take the natural logarithm of both sides.

$$\begin{aligned} \ln e^{5k} &= \ln(1/32) \\ 5k &= -\ln 32 \\ k &= -\frac{1}{5}\ln 32 = -\ln 2 \end{aligned}$$

where we have observed that $32 = 2^5$ but $k = -(\ln 32)/5$ is correct too.

(3) 3. Find the exact value of $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\pi/3$ because $\sin(-\pi/3) = -\sqrt{3}/2$.

(18) 4. Evaluate the limit, if it exists.

(a) $\displaystyle\lim_{x\to 1} \frac{x^2+4x-5}{x^2-1}$

Solution: Plugging $x = 1$ in gives $0/0$ which suggests factoring the top and bottom.

$$\lim_{x\to 1} \frac{x^2+4x-5}{x^2-1} = \lim_{x\to 1} \frac{(x-1)(x+5)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{x+5}{x+1} = \frac{6}{2} = 3$$

(b) $\displaystyle\lim_{h\to 0} \frac{(2+h)^3-8}{h}$

Solution: Plugging in $h = 0$ gives $0/0$ and so we should simplify the numerator.

$$\begin{aligned} \lim_{h\to 0} \frac{(2+h)^3-8}{h} &= \lim_{h\to 0} \frac{8+12h+6h^2+h^3-8}{h} = \lim_{h\to 0} \frac{12h+6h^2+h^3}{h} \\ &= \lim_{h\to 0} \frac{h(12+6h+h^2)}{h} = \lim_{h\to 0} 12+6h+h^2 = 12 \end{aligned}$$

(c) $\displaystyle\lim_{x\to 2+} \frac{2-x}{|2-x|}$

Solution: Again, plugging in $x = 2$ gives $0/0$ so that we must simplify. We need only consider $x > 2$ where $0 > 2-x$ so that $|2-x| = -(2-x)$. Therefore

$$\lim_{x\to 2+} \frac{2-x}{|2-x|} = \lim_{x\to 2+} \frac{2-x}{-(2-x)} = \lim_{x\to 2+} -1 = -1$$

(d) $\displaystyle\lim_{\theta\to 0} \frac{\theta}{\tan 2\theta}$

Solution: Recall that $\lim_{x\to 0}(\sin x)/x = 1$ so that $\lim_{x\to 0}(\tan x)/x = \lim_{x\to 0}(\sin x)/x \lim_{x\to 0}$
$1 \cdot 1 = 1$. Therefore

$$\lim_{\theta\to 0} \frac{\theta}{\tan 2\theta} = \frac{1}{2} \lim_{\theta\to 0} \frac{2\theta}{\tan 2\theta} = \frac{1}{2} 1/ \lim_{x\to 0} \frac{\tan x}{x} = \frac{1}{2}.$$

(7) 5. Determine the infinite limit. $\lim_{t \to -1} \frac{t}{(t+1)^2}$

Solution: This limit is of the form $-1/0$ so that the answer should be ∞, $-\infty$ or "does not exist." By the product rule for limits

$$\lim_{t \to -1} \frac{t}{(t+1)^2} = (\lim_{t \to -1} t \lim_{t \to -1} \frac{1}{(t+1)^2} = - \lim_{t \to -1} \frac{1}{(t+1)^2}$$

so that we need only consider $\lim_{t \to -1} 1/(t+1)^2$ The graph of $y = 1/(t+1)^2$ is that of $y = 1/t^2$ shifted 1 unit left and so $\lim_{t \to -1} 1/(t+1)^2 = \infty$ and

$$\lim_{t \to -1} \frac{t}{(t+1)^2} = -\infty$$

(12) 6. Let $f(x) = \begin{cases} x+2 & \text{if} \quad x \leq -1 \\ -x & \text{if} \quad -1 < x < 1 \\ x+1 & \text{if} \quad x \geq 1 \end{cases}$

(a) Find all points where f is discontinuous.

Solution: The polynomial expressions used to define $f(x)$ are themselves continuous and so the only question is whether f is continuous at $x = -1$ and $x = 1$ where f transits from one rule to another. (Do we have to lift the pencil from the paper when graphing?) Compute the left and right limits at $x = -1$

$$\lim_{x \to -1-} f(x) = \lim_{x \to -1-} x + 2 = -1 + 2 = 1$$

and

$$\lim_{x \to -1+} f(x) = \lim_{x \to -1+} -x = -(-1) = 1$$

and these left and right limits are the same and so f is continuous at $x = -1$. Next consider $x = 1$: Here the left and right limits are

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} -x = -1 = -1$$

and

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} x + 1 = 2 =$$

and so the left and right limits are not the same and so f is not continuous at $x = 1$. In summary f is continuous everywhere but $x = 1$

(b) Sketch the graph of $f(x)$

Solution: The graph consists of 3 straight line segments.

(28) 7. Differentiate the function.

(a) $f(x) = -2x^{11} + \sqrt{x} + x^{-1}$
Solution: Rewrite $\sqrt{x} = x^{1/2}$. Apply the power rule to every term.

$$f'(x) = -2(11)x^{11-1} + \frac{1}{2}x^{1/2-1} + (-1)x^{-1-1} = -22x^{10} + \frac{1}{2}x^{-1/2} - x^{-2}$$

(b) $f(x) = (x^4 + x)\, e^x$
Solution: Apply the product rule

$$\begin{aligned} \frac{d}{dx}f(x) &= (x^4+x)\frac{d}{dx}(e^x) + (e^x)\frac{d}{dx}(x^4+x) \\ &= (x^4+x)(e^x) + (e^x)(4x^3+1) = e^x(x^4+4x^3+x+1) \end{aligned}$$

(c) $h(s) = \dfrac{5s^3 + 8s}{s^4 + 14}$
Solution: Apply the quotient rule.

$$\begin{aligned} \frac{d}{ds}h(s) &= \frac{(s^4+14)\frac{d}{ds}(5s^3+8s) - (5s^3+8s)\frac{d}{ds}(s^4+14)}{(s^4+14)^2} \\ &= \frac{(s^4+14)(15s^2+8) - (5s^3+8s)4s^3}{(s^4+14)^2} \end{aligned}$$

The expression may be simplified (but needn't be) as

$$\frac{d}{ds}h(s) = \frac{-5s^6 - 24s^4 + 210s^2 + 112}{(s^4+14)^2}$$

(11) 8. Find an equation of the tangent line to the curve

$$y = x^{1/2} + x^{-1/2} \qquad \text{at } x = 4.$$

Solution: Find the slope of the tangent line by differentiating using the power rule

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}$$

At $x = 4$, $dy/dx = (1/2)(1/2) - (1/2)(1/8) = 3/16$. The slope is $3/16$ and the line passes through the point on the curve where $x = 4$ and $y = 2 + 1/2 = 5/2$ An equation for the tangent line is

$$y - y_0 = m(x - x_0) \qquad \text{or} \qquad y - \frac{5}{2} = \frac{3}{16}(x - 4)$$

which simplifies to $16y = 3x + 28$.

(12) 9. Find the derivative dy/dx of $y = f(x)$ at $x = 1$ *using the definition of derivative* ($\lim_{h\to 0}(f(x+h) - f(x))/h$) if

$$f(x) = \frac{1}{2x+1}.$$

Solution: Compute the difference quotient $(f(1+h)-f(1))/h$. Since $f(1)=1/3$ and $f(1+h)=1/(2(1+h)+1)=1/(3+2h)$ we have

$$\begin{aligned}
\frac{f(1+h)-f(1)}{h} &= \frac{1}{h}\left[\frac{1}{3+2h}-\frac{1}{3}\right]=\frac{1}{h}\left[\frac{3}{3(3+2h)}-\frac{3+2h}{3(3+2h)}\right] \\
&= \frac{1}{h}\left[\frac{3-(3+2h)}{3(3+2h)}\right] \\
&= \frac{1}{h}\left[\frac{-2h}{3(3+2h)}\right]=\frac{-2}{3(3+2h)}
\end{aligned}$$

(The second equality involved taking a common denominator.) We have managed to cancel the h on the bottom with an h on top and so we are ready to take the limit: the derivative of f at $x=1$ is

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\frac{-2}{3(3+2h)}=-\frac{2}{9}$$