 Test 1B, Math 1850 Section 012
10/2/14 Solutions Name ________________

Materials: Nongraphing calculator; formula sheet. Page 6 provides additional space.

(4) 1. Solve for y in terms of x if $\ln y - \ln 5 = 2x + 1$. Simplify your answer.

Exponentiate both sides

$$e^{\ln y - \ln 5} = e^{2x+1} \quad \text{or} \quad \frac{e^{\ln y}}{e^{\ln 5}} = e^{2x+1}$$

and so $y/5 = e^{2x+1}$ or $y = 5e^{2x+1}$

(4) 2. Simplify the expressions. $\log_2\left(\frac{1}{8}\right) = \log_2(2^{-3}) = -3$

(4) 3. Find the exact value of $\arccos(-1/2)$.

Note $\arccos x$ takes values between 0 and π and so we are looking for the value of θ, $0 \le \theta \le \pi$ so that $\cos\theta = -1/2$ and that is $\theta = 2\pi/3$.

4. Evaluate the limit, if it exists.

(6) (a) $\lim_{t\to -2} \frac{t^2+t-2}{2t+t^2} = \lim_{t\to -2} \frac{(t+2)(t-1)}{t(t+2)} = \lim_{t\to -2} \frac{t-1}{t} = \frac{-3}{-2} = \frac{3}{2}$

(6) (b) $\lim_{h\to 0} \frac{\sqrt{3+h}-\sqrt{3}}{h}$

Here we multiply by the "conjugate."

$$\begin{aligned}
\lim_{h\to 0} \frac{\sqrt{3+h}-\sqrt{2}}{h} &= \lim_{h\to 0} \frac{\sqrt{3+h}-\sqrt{3}}{h} \frac{\sqrt{3+h}+\sqrt{3}}{\sqrt{3+h}+\sqrt{3}} \\
&= \lim_{h\to 0} \frac{(3+h)-3}{h(\sqrt{3+h}+\sqrt{3})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{3+h}+\sqrt{3})} = \lim_{h\to 0} \frac{1}{\sqrt{3+h}+\sqrt{3}} = \frac{1}{\sqrt{3}+\sqrt{3}} = \frac{1}{2\sqrt{3}}
\end{aligned}$$

(6) (c) $\lim_{x\to 2-} 3x\frac{x-2}{|x-2|}$,

When $x < 2$ $|x-2| = -(x-2)$ so that

$$\lim_{x\to 2-} 3x\frac{x-2}{|x-2|} = \lim_{x\to 2-} 3x \lim_{x\to 2-} \frac{x-2}{-(x-2)} 6(-1) = -6$$

(6) (d) $\lim_{\theta\to 0} \frac{3\theta}{\sin 2\theta}$

We recall that $\lim_{x\to 0}(\sin x)/x = 1$ so that $\lim_{x\to 0} x/(\sin x) = 1$ (take reciprocals). If we replace x by 2θ: $\lim_{\theta\to 0} 2\theta/(\sin 2\theta) = 1$. Therefore

$$\lim_{\theta\to 0} \frac{3\theta}{\sin 2\theta} \frac{3}{2} \lim_{\theta\to 0} \frac{2\theta}{\sin 2\theta} = \frac{3}{2}(1) = \frac{3}{2}$$

(6) 5. Find the limit. $\lim_{x\to\infty} \dfrac{3-4x+2x^2}{11x-x^2}$

The limit is of the form ∞/∞ and that suggests we should try to cancel a power of x from top and bottom. We factor out the highest power of x (x^2) from the bottom and the same power from the top.

$$\lim_{x\to\infty} \frac{3-4x+2x^2}{11x-x^2} = \lim_{x\to\infty} \frac{x^2}{x^2}\frac{3/x^2-4/x+2}{11/x-1} = \lim_{x\to\infty} \frac{3/x^2-4/x+2}{11/x-1} = \frac{2}{-1} = -2$$

(9) 6. For what value of a is

$$f(x) = \begin{cases} x^2-5 & \text{if} \quad x<3 \\ ax & \text{if} \quad 3\le x \end{cases}$$

continuous at every x?

The only point where f might not be continuous is $x=3$ which is where f transitions from x^2-5 to ax (both of which are continuous). We need to find a so that $\lim_{x\to 3} f(x)$ exists and is $f(3)=3a$. Compute the left and right sided limits.

$$\lim_{x\to 3-} f(x) = \lim_{x\to 3-} x^2-5 = 4 \quad \text{and} \quad \lim_{x\to 3+} f(x) = \lim_{x\to 3+} ax = 3a$$

For $\lim_{x\to 2} f(x)$ to exist we need the two one sided limits to be equal: $4=3a$ or we need $a=4/3$. If $a=4/3$ then $\lim_{x\to 3} f(x) = 3a = f(3)$ and so $f(x)$ is continuous at $x=3$ and so continuous everywhere. Choose $a=4/3$.

7. Differentiate the function.

(28)

(a) $f(x) = x^3+3+\dfrac{1}{x^3}$

Here $f(x)=x^3+3+x^{-3}$ so that, by the power rule

$$f'(x) = 3x^2-3x^{-4}$$

(b) $g(t) = 3\sqrt{t}+\dfrac{1}{3\sqrt{t}}$

Here $g(t)=3t^{1/2}+(1/3)t^{-1/2}$ so that, by the power rule

$$g'(t) = 3\,\frac{1}{2}t^{-1/2}+\frac{1}{3}\,\frac{-1}{2}t^{-3/2} = \frac{3}{2}t^{-1/2}-\frac{1}{6}t^{-3/2}$$

(c) $g(x)=xe^x$

Apply the product rule: $h'(x)=xe^x+e^x(1)=(x+1)e^x$

(d) $h(t)=\dfrac{3t+5}{t^4+7t}$

The quotient rule is appropriate here.

$$f'(t) = \frac{(t^4+7t)\frac{d}{dt}(3t+5)-(3t+5)\frac{d}{dt}(t^4+7t)}{[t^4+7t]^2} = \frac{(t^4+7t)3-(3t+5)(4t^3+7)}{[t^4+7t]^2}$$

8. Find an equation of the tangent line to the curve $y = \dfrac{8}{4+x^2}$ at (-2,1).

(10)

We need the slope at $x = -2$ and we differentiate using the quotient or reciprocal rule.

$$y' = \frac{8(-2x)}{[4+x^2]^2}$$

and so when $x = -2$, $y' = 1/2$. An equation for the tangent line is $y - y_1 = m(x - x_1)$ (slope point form of the line).

$$y - 1 = \frac{1}{2}(x+2) \text{ or } y = \frac{1}{2}x + 2$$

9. Find the derivative dy/dx of $y = f(x)$ at $x = 1$ *using the definition of derivative* if

(11)

$$f(x) = \frac{1}{3x+1}.$$

Recall the defintion of the derivative: $f'(1) = \lim_{h\to 0}(f(1+h)-f(1))/h$ Therefore

$$\begin{aligned}
f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{3(1+h)+1} - \frac{1}{3(1)+1}\right) \\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{4+3h} - \frac{1}{4}\right) \\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{4-(4+3h)}{(4+3h)4}\right) \\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{-3h}{(4+3h)4}\right) \\
&= \lim_{h\to 0}\frac{-3}{(4+3h)4} = \frac{-3}{16}
\end{aligned}$$

Therefore $f'(1) = -3/16$. This can be checked by the reciprocal rule ($f'(x) = -3/(3x+1)^2$).