 Test 1A, Math 1850 Section 011
10/2/14 Solutions Name ________________

Materials: Nongraphing calculator; formula sheet; Page 6 provides additional space.

(4) 1. Solve for y in terms of x if $\ln y = 2\ln x + 3x$. Simplify your answer.

Exponentiate both sides

$$e^{\ln y} = e^{2\ln x + 3x} \quad \text{or} \quad y = e^{2\ln x}e^{3x} = e^{\ln x^2}e^{3x} = x^2e^{3x}$$

and so $y = x^2e^{3x}$

(4) 2. Simplify the expressions. $\log_3\left(\frac{1}{9}\right) = \log_3 3^{-2} = -2$

(4) 3. Find the exact value of $\arccos(-\sqrt{3}/2) = \arccos(\cos(5\pi/6)) = \frac{5\pi}{6}$.

Note $\arccos x$ takes values between 0 and π and so we are looking for the value of θ, $0 \le \theta \le \pi$ so that $\cos\theta = -\sqrt{3}/2$ and that is $\theta = 5\pi/6$.

4. Evaluate the limit, if it exists.

(6) (a) $\lim_{t\to 3} \frac{t^2 - 2t - 3}{3t - t^2} = \lim_{t\to 3} \frac{(t-3)(t+1)}{-t(t-3)} = \lim_{t\to 3} -\frac{t+1}{t} = -\frac{4}{3}$

(6) (b) $\lim_{h\to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h}$

Here we multiply by the "conjugate."

$$\begin{aligned}
\lim_{h\to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h} &= \lim_{h\to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h} \frac{\sqrt{2+h} + \sqrt{2}}{\sqrt{2+h} + \sqrt{2}} \\
&= \lim_{h\to 0} \frac{(2+h) - 2}{h(\sqrt{2+h} + \sqrt{2})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{2+h} + \sqrt{2})} = \lim_{h\to 0} \frac{1}{\sqrt{2+h} + \sqrt{2}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}
\end{aligned}$$

(6) (c) $\lim_{x\to 1-} 2x\frac{x-1}{|x-1|}$,

When $x < 1$ $|x-1| = -(x-1)$ so that

$$\lim_{x\to 1-} 2x\frac{x-1}{|x-1|} = \lim_{x\to 1-} 2x \lim_{x\to 1-} \frac{x-1}{-(x-1)} 2(-1) = -2$$

(6) (d) $\lim_{\theta\to 0} \frac{2\theta}{\sin 5\theta}$

We recall that $\lim_{x\to 0}(\sin x)/x = 1$ so that $\lim_{x\to 0} x/(\sin x) = 1$ (take reciprocals). If we replace x by 5θ: $\lim_{\theta\to 0} 5\theta/(\sin 5\theta) = 1$. Therefore

$$\lim_{\theta\to 0} \frac{2\theta}{\sin 5\theta} \frac{2}{5} \lim_{\theta\to 0} \frac{5\theta}{\sin 5\theta} = \frac{2}{5}(1) = \frac{2}{5}$$

(6) 5. Find the limit. $\lim_{x\to\infty} \dfrac{3-4x+2x^2}{4-x^2}$

The limit is of the form ∞/∞ and that suggests we should try to cancel a power of x from top and bottom. We factor out the highest power of x (x^2) from the bottom and the same power from the top.

$$\lim_{x\to\infty} \frac{3-4x+2x^2}{4-x^2} = \lim_{x\to\infty} \frac{x^2}{x^2}\frac{3/x^2-4/x+2}{4/x^2-1} = \lim_{x\to\infty} \frac{3/x^2-4/x+2}{4/x^2-1} = \frac{2}{-1} = -2$$

(9) 6. For what value of a is

$$f(x) = \begin{cases} x^2-1 & \text{if } x<2 \\ ax & \text{if } 2\le x \end{cases}$$

continuous at every x?

The only point where f might not be continuous is $x = 2$ which is where f transitions from $x^2 - 1$ to ax (both of which are continuous). We need to find a so that $\lim_{x\to 2} f(x)$ exists and is $f(2) = 2a$. Compute the left and right sided limits.

$$\lim_{x\to 2-} f(x) = \lim_{x\to 2-} x^2-1 = 3 \quad\text{and}\quad \lim_{x\to 2+} f(x) = \lim_{x\to 2+} ax = 2a$$

For $\lim_{x\to 2} f(x)$ to exist we need the two one sided limits to be equal: $3 = 2a$ or we need $a = 3/2$. If $a = 3/2$ then $\lim_{x\to 2} f(x) = 2a = f(2)$ and so $f(x)$ is continuous at $x = 2$ and so continuous everywhere. Choose $a = 3/2$.

7. Differentiate the function.

(28)

(a) $f(x) = x^2 + 2 + \dfrac{1}{x^2}$

Here $f(x) = x^2 + 2 + x^{-2}$ so that, by the power rule

$$f'(x) = 2x - 2x^{-3}$$

(b) $g(t) = 2\sqrt{t} + \dfrac{1}{2\sqrt{t}}$

Here $g(t) = 2t^{1/2} + (1/2)t^{-1/2}$ so that, by the power rule

$$g'(t) = 2\,\frac{1}{2}t^{-1/2} + \frac{1}{2}\,\frac{-1}{2}t^{-3/2} = t^{-1/2} - \frac{1}{4}t^{-3/2}$$

(c) $h(x) = x^2\, e^x$

Apply the product rule: $h'(x) = x^2e^x + e^x(2x) = (x^2 + 2x)e^x$.

(d) $f(t) = \dfrac{5t+7}{t^5+3t}$

The quotient rule is appropriate here.

$$f'(t) = \frac{(t^5+3t)\frac{d}{dt}(5t+7) - (5t+7)\frac{d}{dt}(t^5+3t)}{[t^5+3t]^2} = \frac{(t^5+3t)5 - (5t+7)(5t^4+3)}{[t^5+3t]^2}$$

8. Find an equation of the tangent line to the curve $y = \dfrac{9}{5+x^2}$ at (2,1).

(10) We need the slope at $x = 2$ and we differentiate using the quotient or reciprocal rule.

$$y' = \frac{9(-2x)}{[5+x^2]^2}$$

and so when $x = 2$, $y' = -4/9$. An equation for the tangent line is $y - y_1 = m(x - x_1)$ (slope point form of the line).

$$y - 1 = -\frac{4}{9}(x-2) \text{ or } y = \frac{-4x+17}{9}$$

9. Find the derivative dy/dx of $y = f(x)$ at $x = 1$ *using the definition of derivative* if

(11)
$$f(x) = \frac{1}{2x+3}.$$

Recall the defintion of the derivative: $f'(1) = \lim_{h\to 0}(f(1+h) - f(1))/h$ Therefore

$$\begin{aligned} f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{2(1+h)+3} - \frac{1}{2(1)+3}\right) \\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{5+2h} - \frac{1}{5}\right) \\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{5-(5+2h)}{(5+2h)5}\right) \\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{-2h}{(5+2h)5}\right) \\ &= \lim_{h\to 0}\frac{-2}{(5+2h)5} = \frac{-2}{25} \end{aligned}$$

Therefore $f'(1) = -2/25$. This can be checked by the reciprocal rule ($f'(x) = -2/(2x+3)^2$).