 Final Exam Section 1,2,3,4,9 (circle one)
12-13-06 Math 1850 Name ____________________

Justify your work. A nongraphing calculator is permitted but not needed. The exam is 2 hours and 200 points are possible; the value of each question is indicated in the left margin. There are 9 pages, including this one: check that no pages are missing.
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(24) 1. Differentiate the function.

(a) $f(x) = \dfrac{2x+1}{x^2-x}$
Solution: By the quotient rule

$$f'(x) = \frac{2(x^2-x)-(2x+1)(2x-1)}{(x^2-x)^2} = \frac{-2x^2-2x+1}{(x^2-x)^2}$$

(b) $h(x) = 5x^3 \sin x$
Solution: By the product rule.

$$h'(x) = 5x^3 \cos x + 15x^2 \sin x$$

(c) $g(x) = \dfrac{1}{(x^4+3x)^{2/3}}$
Solution: Here $g(x) = (x^4+3x)^{-2/3}$ so that, by the generalized power rule

$$g'(x) = -\frac{2}{3}(x^4+3x)^{-5/3}(4x^3+3)$$

(d) $f(\theta) = \csc(8\theta)$
Solution: By the chain rule $f'(\theta) = -\csc(8\theta)\cot(8\theta)8 = -8\csc(8\theta)\cot(8\theta)$.

(15) 2. Evaluate the limit, if it exists.

(a) $\displaystyle\lim_{x\to 4} \frac{3x-12}{x^2-x-12}$
Solution: This limit is of the form $0/0$ and so we factor

$$\lim_{x\to 4} \frac{3x-12}{x^2-x-12} = \lim_{x\to 4} \frac{3(x-4)}{(x-4)(x+3)} = \lim_{x\to 4} \frac{3}{x+3} = \frac{3}{7}.$$

(b) $\displaystyle\lim_{\theta\to 0} \frac{\tan(4\theta)}{\theta}$
Solution: We know

$$\lim_{\theta\to 0} \frac{\tan(4\theta)}{\theta} = 4\lim_{\theta\to 0} \frac{\sin 4\theta}{4\theta}\left(\frac{1}{\cos 4\theta}\right) = 4(1)(1) = 4$$

because $\lim_{x\to 0}(\sin x)/x = 1$ and of course $\cos 0 = 1$.

(c) $\displaystyle\lim_{x\to\infty} \frac{-6x+2x^3}{4-x^2-5x^3}$
Solution: This limit is of the form ∞/∞ and so we factor out the highest power from the bottom.

$$\lim_{x\to\infty} \frac{-6x+2x^3}{4-x^2-5x^3} = \lim_{x\to\infty} \left(\frac{x^3}{x^3}\right)\frac{-6/x^2+2}{4/x^3-1/x-5} = \lim_{x\to\infty} \frac{-6/x^2+2}{4/x^3-1/x-5} = -\frac{2}{5}$$

(9) 3. Find dy/dx by implicit differentiation. $x^2 - 5xy + \sqrt{y} = -10$

Solution: This equation defines y implicitly as a function of x and so $y = y(x)$. Differentiate in x.

$$2x - 5(xy' + y) + \frac{1}{2}y^{-1/2}y' = 0 \text{ so that } 2x - 5y = (5x - \frac{1}{2}y^{-1/2})y'$$

Solve for y'

$$y' = \frac{2x - 5y}{5x - \frac{1}{2}y^{-1/2}} = \frac{4xy^{1/2} - 10y^{3/2}}{10xy^{1/2} - 1}$$

(12) 4. Find all numbers at which the function

$$f(x) = \begin{cases} 2x + 3 & \text{if} \quad x \leq 0 \\ 2 - x & \text{if} \quad 0 < x < 2 \\ (x-2)^2 & \text{if} \quad x \geq 2 \end{cases}$$

is discontinuous. Explain your answer using, for example, limits or a sketch of the graph.

Solution: The defining relations are all continuous because they are polynomials on the intervals $x < 0$, $0 < x < 2$ and $x > 2$. We must just check the two transitions points $x = 0$ and $x = 2$. First we check the limits at these two points

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} 2x + 3 = 3 \quad \text{and} \quad \lim_{x \to 0+} f(x) = \lim_{x \to 0+} 2 - x = 2$$
$$\lim_{x \to 2-} f(x) = \lim_{x \to 2-} 2 - x = 0 \quad \text{and} \quad \lim_{x \to 2+} f(x) = \lim_{x \to 2+} (x-2)^2 = 0$$

Since the limits from the left and right at $x = 0$ are not equal, the functions is not continuous at $x = 0$. However at $x = 2$ the limits from the left and right are equal and so the the limit does exist. $\lim_{x \to 2} f(x) = 0$ and of course $f(2) = 0$ and so this shows f is continuous at $x = 2$. In summary f is continous everywhere but $x = 0$. The sketch confirms that the only break in the graph is at $x = 0$.

(9) 5. Find an equation for the tangent line to the curve $y = \sqrt{8 + (1/x)}$ at (1,3).

Solution: Differentiate $y = (8 + x^{-1})^{1/2}$ to get $y' = 1/2(8 + x^{-1})^{-1/2}(-x^{-2})$. When $x = 1$, $y' = -1/6$ so the an equation for the tangent line is

$$y - 3 = -\frac{1}{6}(x - 1) \quad \text{or} \quad y = -\frac{1}{6}x + \frac{19}{6}$$

(11) 6. Use *the definition of the derivative* to find the derivative of the function.

$$f(x) = \frac{2}{3x - 1}.$$

Solution: We want to compute $\lim_{h\to 0}(f(x+h)-f(x))/h$. Consider first

$$\begin{aligned} \frac{1}{h}(f(x+h)-f(x)) &= \frac{1}{h}\left(\frac{2}{3(x+h)-1}-\frac{2}{3x-1}\right) \\ &= \frac{1}{h}\left(\frac{2(3x-1)-2(3(x+h)-1)}{(3(x+h)-1)(3x-1)}\right) \\ &= \frac{1}{h}\left(\frac{-6h}{(3(x+h)-1)(3x-1)}\right) = \left(\frac{-6}{(3(x+h)-1)(3x-1)}\right) \end{aligned}$$

Therefore

$$\lim_{h\to 0}\frac{1}{h}(f(x+h)-f(x)) = \lim_{h\to 0}\frac{-6}{(3(x+h)-1)(3x-1)} = \frac{-6}{(3x-1)^2}$$

so that $f'(x) = -6(3x-1)^{-2}$ and this could be checked by the generalized power rule.

(10) 7. Evaluate the definite integral $\int_0^{\pi/2}\left(x\sqrt{x}+4\cos x\right)\,dx$.

Solution: Compute the antiderivative.

$$\int_0^{\pi/2}\left(x\sqrt{x}+4\cos x\right)\,dx = \int_0^{\pi/2}\left(x^{3/2}+4\cos x\right)\,dx = \frac{2}{5}x^{5/2}+4\sin x\Big|_0^{\pi/2} = \frac{2}{5}\left(\frac{\pi}{2}\right)^{5/2}+4$$

(18) 8. Evaluate the indefinite integral.

(a) $\displaystyle\int \frac{5t+t^{1/3}}{t^3}\,dt$

Solution: Simplify the integrand.

$$\int \frac{5t+t^{1/3}}{t^3}\,dt = \int 5t^{-2}+t^{-8/3}\,dt = 5\frac{1}{-1}t^{-1}-\frac{3}{5}t^{-5/3}+C = -5t^{-1}-\frac{3}{5}t^{-5/3}+C$$

and we check by differentiating $-5t^{-1}-\frac{3}{5}t^{-5/3}$ to get $5t^{-2}-\frac{3}{5}(-\frac{5}{3})t^{-8/3} = 5t^{-2}+t^{-8/3}$. It checks.

(b) $\displaystyle\int t^2\sqrt{5+t^3}\,dt$

Solution: Substitute $u = 5+t^3$ so that $du = 3t^2\,dt$ or $du/3 = t^2\,dt$. Therefore

$$\int t^2\sqrt{5+t^3}\,dt = \frac{1}{3}\int\sqrt{u}\,du = \frac{1}{3}\int u^{1/2}\,du = \frac{2}{9}u^{3/2}+C = \frac{2}{9}(5+t^3)^{3/2}$$

Check by differentiation: the derivative of $\frac{2}{9}(5+t^3)^{3/2}$ is $\frac{2}{9}\frac{3}{2}(5+t^3)^{1/2}3t^2 = t^2(5+t^3)^{1/2}$ by the chain rule. It checks.

(12) 9. Find a function $f(x)$ such that $f'(x) = 3x+\sec^2 x$ and $f(1) = 1/2$.

Solution: Using antiderivatives we find $f(x) = (3/2)x^2+\tan x + C$. Check by differentiation. Since we further know $1/2 = f(1) = (3/2)1^2+\tan(1)+C$ we see that $1/2 = 3/2+\pi/4+C$ so that $C = -1-\pi/4$ and

$$f(x) = (3/2)x^2+\tan x - 1 - \pi/4$$

(8) 10. Find the *derivative* of the function $g(x) = \int_2^x \frac{t}{t^4+7}\,dt$.

Solution: By the Fundamental Theorem of Calculus:

$$g'(x) = \frac{x}{x^4+7}$$

11. Find the absolute maximum and minimum of $f(x) = 4x^2 - x^4$, on the closed interval $-2 \leq x \leq 3$.

(14)

Solution: We use the closed interval method.

(a) We find all critical numbers. Since $f'(x) = 8x - 4x^3 = 4x(2 - x^2)$ we have critical points at $x = 0$, $x = \sqrt{2}$ and $x = -\sqrt{2}$. Since f' is defined everywhere, there are no other critical numbers.

(b) The endpoints of the interval are $x = -2$ and $x = 3$.

(c) Evaluate f at each of the numbers found above. $f(-2) = 0$, $f(-\sqrt{2}) = 4$, $f(0) = 0$ and $f(\sqrt{2}) = 4$ and $f(3) = -45$. Therefore the absolute maximum value is 4 and it occurs at two places $x = \pm\sqrt{2}$ and the absolute minimum is -45 and it occurs at the endpoint $x = 3$.

(12) 12. Sketch the graph of a function that satisfies all of the given conditions.

$$\begin{array}{lll} f'(-2) = 0; f'(0) = 0; & & \\ f'(x) < 0 & \text{if} & x < -2 \text{ or } x > 0, \\ f'(x) > 0 & \text{if} & -2 < x < 0, \\ f''(x) > 0 & \text{if} & x < -1 \text{ or } x > 1, \\ f''(x) < 0 & \text{if} & -1 < x < 1. \end{array}$$

Solution: The sign conditions on the derivative allow us to determine the intervals of increase and decrease.

Interval	Evaluate f'	Increasing or Decreasing
$x < -2$	$f' < 0$	Decreasing
$-2 < x < 0$	$f' > 0$	Increasing
$x > 0$	$f' < 0$	Decreasing

We further know that $x = -2$ and $x = 0$ are critical numbers and the first derivative test tells us that $x = -2$ is a local min and $x = 0$ is a local max. We also know from the sign conditions on the second derivative the intervals of concavity.

Interval	Evaluate y''	Concavity
$x < -1$	$f'' > 0$	Up
$-1 < x < 1$	$''(-1/2) < 0$	Down
$1 < x$	$f'' > 0$	up

The concavity helps us determine the shape of the graph and it determines that although f is decreasing for $x > 0$ the rate of decrease diminshes after $x = 1$.

(18) 13. Let $f(x) = \dfrac{x-1}{x^2}$

(a) Find the vertical and horizontal aymptotes, if any.
(b) Find the intervals of increase or decrease.
(c) Find the local maximum and minimum values.
(d) Use the above information to sketch the graph $y = f(x)$.

Solution: There is a vertical asymptote at $x = 0$ because there is a division by 0 there. It follows that $x = 0$ is not in the domain of f. For horizontal asymptotes we consider

$$\lim_{x\to\infty} \frac{x-1}{x^2} = \lim_{x\to\infty} \frac{1}{x} - \frac{1}{x^2} = 0 - 0 = 0.$$

Therefore $y = 0$ is a horizontal asymptote as $x \to \infty$ and, by a similar calculation, as $x \to -\infty$.
Compute the derivative:

$$f'(x) = \frac{x^2 - 2x(x-1)}{x^4} = \frac{2-x}{x^3}$$

Set $f'(x) = 0$ and we see $x = 2$ is a critical number. Since f' is defined everywhere f is, that is where $x \neq 0$ this is the only critical point.

Interval	Evaluate f'	Increasing or Decreasing
$x < 0$	$f'(-1) = -3 < 0$	Decreasing
$0 < x < 2$	$f'(1) = 1 > 0$	Increasing
$x > 2$	$f'(3) = -1/27$	Decreasing

From which we see the critical number $x = 2$ is a local maximum where $f(2) = 1/4$; there are no local minima. We can graph from this information but of course the second derivative would help get the shape right and check the other computations. We can check $f'' = (2x-6)/x^4$. Since $f''(3) = 0$ we see (3,2/9) is an inflection point.

Interval	Evaluate y''	Concavity
$x < 0$	$f''(-1) < 0$	Down
$0 < x < 3$	$f''(1) < 0$	Down
$3 < x$	$f''(4) > 0$	Up

We may now graph.

14. Two cars start from the same point. One car leaves at noon and travels west at 60 km/h. The other car leaves at 1 pm and travels south at 50 km/h. At what rate is the distance between
(14) the cars increasing at 2 pm?

Solution Let $x(t)$ be the distance that the first car has traveled west, t hours after noon; let $y(t)$ be the distance the second car has traveled. We know that $x' = 60$ and $y'(t) = 50$ if $t > 1$. Since the cars are traveling on straight paths perpendicular to each other the Pythagorean theorem says that the distance, $z(t)$ say, between them is $z^2 = (x^2 + y^2)$. We want z'. Differentiate in t:

$$2zz' = 2xx' + 2yy' \quad \text{or} \quad zz' = xx' + yy'$$

When $t = 2$ we have $x = 120$ (60 km/h for two hours) and $y = 50$ (50 km/h for one hour) and $z^2 = 120^2 + 50^2$ so that $z = 130$. Therefore $130z' = 120(60) + 50(50)$ so that $z' = 970/13 \approx 74.6$. The cars are separating at 970/13 km/h.

Solution 2: An alternative approach is to use Cartesian coordinates: The first car's position is (-60t,0) and the second car's position is $(0, -50(t-1))$ if $t > 1$ where the starting point of the cars is the origin and west is down the negative x-axis and south is down the negative y-axis. Then the distance between the cars is $z^2 = ((-60t)^2 + (-50(t-1))^2)$. We want $z'(2)$ and $2zz' = 60^2(2t) + 50^2(2t-2)$ so that when $t = 2$, $(130)z' = 60^2(2) + 50^2$ so that $z' = 970/13$.

15. If 7200 in^2 of material are available to make a box with square base and open top, find the
(14) largest possible volume of the box.

Solution: Let the sidelength of the square base be x and the height of the box be y. We want to maximize the volume $V = x^2y$ (length $\times$ width $\times$ height). However there is only 7200 in^2 of material available and that means that $x^2 + 4xy = 7200$. Therefore $y = 1800/x - x/4$ and $V = 1800x - x^3/4$. We want to maximize V for $x > 0$. Differentiate $V' = 1800 - 3x^2/4$. Set $V' = 0$ to check for critical numbers. $1800 = 3x^2/4$ or $x = \sqrt{2400}$. Since V' is defined everywhere this is the only positive critical number. Perform the first derivative test. If $0 < x < \sqrt{2400}$ then $V'(x) > 0$ because $V'(1) = 1800 - 3/4 > 0$. If $\sqrt{2400} < x$ then $V' < 0$ because $V'(50) = -75 < 0$ so that V increases on $0 < x < \sqrt{2400}$ and decreases on $\sqrt{2400} < x$ and therefore $x = \sqrt{2400}$ is the absolute maximum possible number corresponding to an absolute maximum volume $V = 1800\sqrt{2400} - (\sqrt{2400})^3/4 = 1200\sqrt{2400} \approx 58,787\text{in}^3$.