 Test 2A, Math 1730-931 Photo ID Check
7/19/13 Solutions Name ____________________

There are five pages, printed two sided. A single formula sheet and *non-graphing* calculator are permitted. There are 100 possible points. One hour is allowed.

1. Calculate the following derivatives. (5 ea)

   (a) $f(x) = \dfrac{1}{\sqrt{4+e^x}} = (4+e^x)^{-1/2}$

   By the extended power rule (or chain rule) $f'(x) = -\frac{1}{2}(4+e^x)^{-3/2}e^x = -\dfrac{e^x}{2}(4+e^x)^{-3/2}$

   (b) $f(x) = x^2 \ln(x+2)$

   By the product rule $f'(x) = x^2\dfrac{1}{x+2} + \ln(x+2)2x = \dfrac{x^2}{x+2} + 2x\ln(x+2)$

   (c) $f(x) = \dfrac{e^{2x}}{5+x}$

   The quotient rule applies here $f'(x) = \dfrac{(5+x)e^{2x}2 - e^{2x}}{(5+x)^2} = \dfrac{e^{2x}[9+2x]}{(5+x)^2}$

2. Calculate the second derivative of $f(x) = \ln(1+x^2)$. (9)

   The first derivative is $f'(x) = \dfrac{1}{1+x^2}2x = \dfrac{2x}{1+x^2}$ by the chain rule. Therefore, by the quotient rule

$$f''(x) = \frac{(1+x^2)2 - 2x(2x)}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}$$

3. Find the absolute maximum and minimum of $f(x) = 4x^2 - \dfrac{1}{3}x^3$ on [0,12]. (Show your work.) (12)

   Here f is a continuous function on a closed and bounded interval and so that the max-min principle 1 applies (page 251). Differentiate $f'(x) = 8x - x^2 = x(8-x)$ so there is a critical point (set $f'(x) = 0$) at $x = 0$ and $x = 8$. Those are the only two critical points because $f'(x)$ is defined everywhere. In addition to the critical points there are the two endpoints $x = 0$ and $x = 12$. Evaluate f at each of the points: $f(0) = 0$; $f(8) = 256/3$; $f(12) = 0$. Therefore the absolute maximum value of f is 256/3 taken at $x = 8$ and the absolute minimum value of 0 is taken at $x = 0$ and $x = 12$.

4. Given the function $f(x) = x^3 - 3x^2 + 2$, (18)

   (a) Find

   - x-values of any critical value(s):
   - x-values of any relative minimum(s):
   - x-values of any relative maximum(s):

- region(s) where f is increasing
- region(s) where f is decreasing

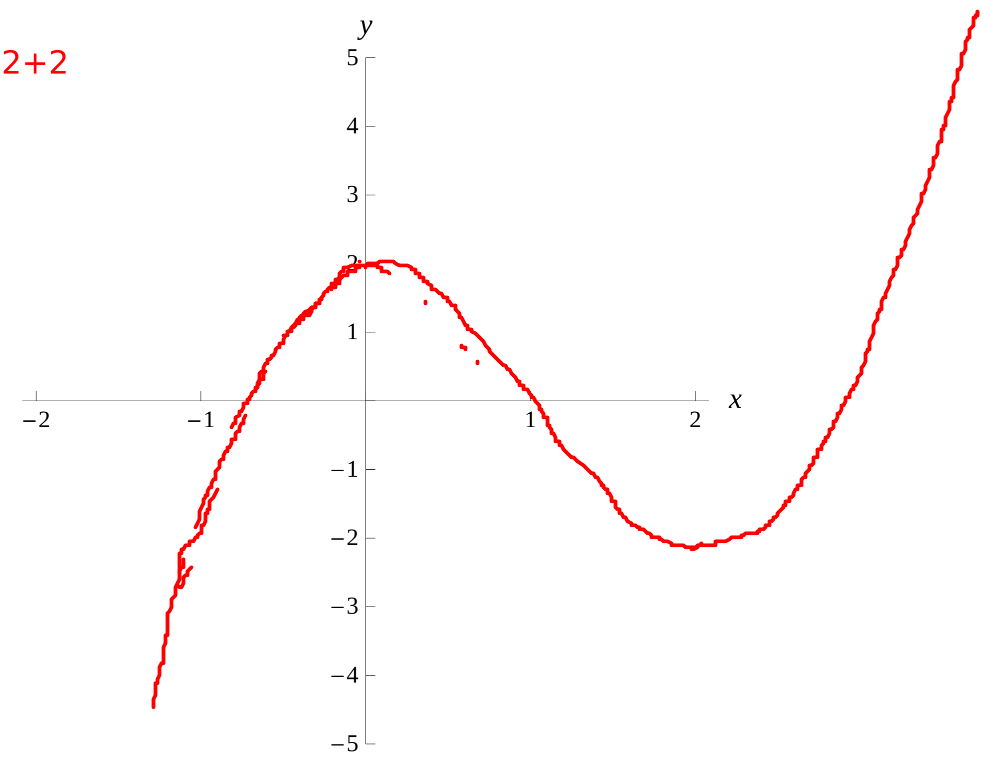
(b) Sketch the graph of $f(x)$.

Differentiate $f'(x) = 3x^2 - 6x = 3x(x-2)$ so that $x = 0$ and $x = 2$ are critical points (where $f'(x) = 0$) and those are the only critical points because $f'(x)$ is defined everywhere. Check for increasing and decreasing regions.

Interval	Evaluate f'	Increasing or Decreasing
$x < 0$	$f'(-1) = 9 > 0$	Incr
$0 < x < 2$	$f'(1) = -3 < 0$	Decr
$2 < x$	$f'(3) = 9 > 0$	Incr

Check that values of f: $f(-1) = -2$; $f(0) = 2$;$f(1) = 0$ and $f(2) = -2$. By the first derivative test, we see $x = 0$ corresponds to a relative max and $x = 2$ correpsonds to a relative min. (There are no other relative extrema.) Now graph.

y = x^3-3x^2+2



(12) 5. If $f(x) = 2x^3 - x^4$ determine

- the x values of any point(s) of inflection
- the region(s) where f is concave up
- the region(s) where f is concave down

Calculate $f'(x) = 6x^2 - 4x^3$; $f''(x) = 12x - 12x^2 = 12x(1-x)$. The points of inflection correspond to values of x where $f''(x) = 0$ and so $x = 0$ and $x = 1$. Now determine the intervals of cancavity.

Interval	Evaluate f''	Concave Up or Down
$x < 0$	$f''(-1) < 0$	Down
$0 < x < 1$	$f''(1/2) > 0$	Up
$1 < x$	$f''(2) < 0$	Down

6. Given $f(x) = \dfrac{x-1}{x^2 - 2x - 3}$ (10)

   (a) Find all vertical asymptotes.

   (b) Find all horizontal asymptotes.

   There are vertical asymptotes where there is a division by 0 and since $x^2 - 2x - 3) = (x-3)(x+1)$ the vertical asympotes are $x = -1$ and $x = 3$. (Note "-1" and "3" is not the correct answer since asymptotes are lines and should be specified by an equation.) For the horizontal asymptote consider

   $$\lim_{x\to\infty} \frac{x-1}{x^2 - 2x - 3} = \lim_{x\to\infty} \frac{x^2}{x^2} \frac{1/x - 1/x^2}{1 - 2/x - 3/x^2} = \lim_{x\to\infty} \frac{1/x - 1/x^2}{1 - 2/x - 3/x^2} = \frac{0}{1} = 0$$

   so that $y = 0$ is the horizontal asymptote at $x = \infty$ and the same calculation shows that $\lim_{x\to-\infty} f(x) = 0$ and so $y = 0$ is the asymptote at $x = -\infty$ as well.

7. A department store sells TV's. Marketing research has determined that if the price of a certain model is $p = 1100 - 4x$ they can sell x units. The department store has costs

   $$C(x) = 3000 + 220x$$

   to obtain x units. (a) Determine the revenue and the profit. (13)

   The revenue is $R(x) = xp = x(1100 - 4x)$ (price times the number sold) and the profit is $P(x) = R(x) - C(x) = x(1100 - 4x) - 3000 - 220x = -3000 + 880x - 4x^2$. (b) Find the price that maximizes the profit.

   Maximize profit: $P'(x) = 880 - 8x$. There is a critical point at $x = 110$ and it is a relative maximum because $P''(110) = -8 < 0$ and so it is an absolute max by max-min Principle 2. The store should sell 110 units and that requires the price $1100 - 4(110) = 660$. The television should be sold for \$ 660 to maximize profit.

8. An initial investment of \$100,000 in a bond fund earns interest compounded continuously at a rate of 3.2% which means that the balance P grows at the rate (11)

   $$\frac{dP}{dt} = 0.032P$$

   (a) Find the value of the investment after 13 years.
   $P(t) = P_0 A e^{kt}$ where $k = 0.032$ and $P_0 = 100,000$ so that $P(t) = 100000e^{0.032t}$. The value after 13 years is $P(t) = 100,000e^{0.032(13)} = 100,000e^{0.416} \approx 151,588.58$ or \$ 151,588.58 . ($100,000e^{0.416}$ is also a correct answer.)

(b) How long will it take the value of the account to double?
We want the time t when $P(t) = 200,000$ or $100,000e^{0.032t} = 200,000$. Divide by 100,000 and take the natural log of both sides: $0.032t = \ln 2$ or

$$t = \frac{\ln 2}{0.032} \quad \text{or} \quad t \approx 21.66$$

It takes about 21 and 2/3 years or 21 years and 8 months to double in value.