5.3 Improper Integrals: Define the improper integral:

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$$

or

$$\int_{-\infty}^{d} f(x)\,dx = \lim_{c \to -\infty} \int_c^d f(x)\,dx$$

Example. The marginal profit for selling x thousand gallons of Poopsi is

$$P'(x) = \frac{1}{x}$$ in hundreds of dollars per thousand gallon s

for $x>1$

What is the profit for selling the 99 thousand gallons after the first 1000 gallons

that is sales $1 < x < 100$

$$P(100)-P(1) = \int_1^{100} \frac{1}{x}\,dx$$

$$= \ln x \Big|_1^{100} = \ln 100 - \ln 1 = \ln 100 = 2 \ln 10$$

What is the profit from sales $1<x<b$?

$$P(b) - P(1) = \int_1^b \frac{1}{x}\,dx = \ln x \Big|_1^b = \ln b - \ln 1 = \ln b$$

The profit in the limit as more and more gallons are sold is (after the first 1000)

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b \to \infty} \int_1^b \frac{1}{x}\,dx$$

$$= \lim_{b \to \infty} \ln x \Big|_1^b = \lim \ \ln b - \cancel{\ln 1} = \infty$$

(Note $\ln 2^n = n \ln 2 \approx n(0.693) \longrightarrow \infty$ as $n \longrightarrow \infty$

so that $\lim_{b \to \infty} \ln b = \infty$

Example: Now suppose that the marginal profit is $P'(x) = \frac{1}{x^2} = x^{-2}$

in hundreds of dollars per thousand gallons after the first thousand $(x>1)$

Then

$$\int_1^{\infty} P'(x)\,dx = \lim_{b \to \infty} \int_1^b x^{-2}\,dx$$

$$= \lim_{b \to \infty} -x^{-1} \Big|_1^b = \lim_{b \to \infty} -b^{-1} - (-1)^{-1}$$

$$= \lim_{b \to \infty} -\frac{1}{b} + 1$$

$$=1$$

Area ▧ = 1

Review time

$$= \int 4x^{(-2)} - 7x^{(-1)}\,dx$$

$$= 4\int x^{(-2)}dx - 7\int x^{(-1)}\,dx$$

$$= 4\,\frac{1}{-1}\,x^{-1} - 7\ln x + C$$

$$= -4x^{-1} - 7\ln x + C$$

$$\int (x^2)^{1/3} - x^{1/2}\,dx$$

$$\approx \int x^{2/3} - x^{1/2}\,dx$$

$$= \frac{1}{5/3}x^{5/3} - \frac{1}{3/2}x^{3/2} + C$$

$$= \frac{3}{5}x^{5/3} - \frac{2}{3}x^{3/2} + C$$

Check by differentiation:

$$\frac{d}{dx}\,\frac{3}{5}x^{5/3} - \frac{2}{3}x^{3/2} + C = \cancel{\frac{3}{5}}\,\cancel{\frac{5}{3}}\,x^{2/3} - \cancel{\frac{2}{3}}\,\cancel{\frac{3}{2}}\,x^{1/2}$$

$$= x^{2/3} - x^{1/2}$$

$f(x) = x^2 \ln(3x)$

Product rule $\quad f'(x) = x^2 \frac{d}{dx} \ln(3x) + \ln(3x) \frac{d}{dx} x^2$

Aside: $\frac{d}{dx} \ln(3x) = \frac{1}{\cancel{3}x} \cancel{3} = \frac{1}{x}$

$\ln(3x) = \ln 3 + \ln x$

$\frac{d}{dx} \ln(3x) = \frac{d}{dx} \ln 3 + \frac{d}{dx} \ln x = 0 + \frac{1}{x}$

$f'(x) = x^2 \frac{1}{x} + \ln(3x)\, 2x = x + 2x \ln(3x)$

23 f) $f(x) = \frac{e^{4x}}{x^4}$

Quotient rule: $\frac{DN' - ND'}{D^2}$

$f'(x) = \frac{x^4 \left(\frac{d}{dx} e^{4x}\right) - e^{4x}\, 4x^3}{(x^4)^2}$

$\frac{d}{dx} e^{4x} = e^{4x}\, 4 = 4 e^{4x}$

$f'(x) = \frac{x^4 (4e^{4x}) - 4x^3 e^{4x}}{x^8}$

$= \frac{4 \cancel{x^3} e^{4x} (x-1)}{x^{\cancel{8}\,5}}$

$= \frac{4 e^{4x} (x-1)}{x^5}$

8 (REview Sheet) Find an equation for the tangent line to $f(x) = x^3 - 2x^2 + 5x$

at (2,10)

$f'(x) = 3x^2 - 2(2x) + 5 = 3x^2 - 4x + 5$

$f'(2) = 3 \cdot 2^2 - 4(2) + 5 = 9$

$y - y_0 = m(x - x_0)$

$y - 10 = 9(x-2)$

$y = 9x - 8$

26 b f9x) = $\log_9 x = f(x)$

26 b $f(x) = \log_9 x = \dfrac{\ln x}{\ln 9}$

$f'(x) = \dfrac{1}{\ln 9} \cdot \dfrac{1}{x}$

26 d) $f(x) = 5\log_3 (x^4 - 7x)$

$f'(x) = 5 \dfrac{1}{x^4 - 7x} \cdot (4x^3 - 7) \dfrac{1}{\ln 3}$

$= \dfrac{5(4x^3 - 7)}{(\ln 3)(x^4 - 7x)}$

20 (Review Sheet)

$R(x) = 2x$

$C(x) = 0.01x^2 + 0.6x + 30$

Maximize profit

Profit is $P(x) = R(x) - C(x) = 2x - (0.01x^2 + 0.6x + 30)$

$= -30 + 1.4x - 0.01x^2$

$P'(x) = 1.4 - 0.02x$

Set $P'(x) = 0$ $\quad 1.4 - 0.02x = 0$ or $1.4 = 0.02x$ $\quad \frac{1.4}{0.02} = x \quad x = 70$

That is the only critical point because $P'(x)$ is defined everywhere.

Check whether $x=70$ corresponds to a relative max or min.

Second Derivative Test $P''(x) = -0.02 < 0$ and so $x=70$ is a relative max.

So its an absolute max

What is the max profit? $P(70) = -30 + 1.4(70) - 0.01(70)^2 = 19$

21 (Review Sheet)

Can sell 20 bicycles per week at \$400 each and for each \$10 reduction in price she can sell 2 more bikes. Cost is \$200 per bicycle. Maximize profit.

price = p = 400 - 10 x

sales 20+2x (number of bikes sold)

R(x) = (20+2x)(400-10x)
C(x) = (20+2x)200
P(x) = R(x) - C(x)
= (20+2x)(400-10x) -(20+2x) 200
= (20+2x) (400-10x -200)
= (20+2x)(200-10x)
= 4000 +200x -20x^2

P'(x) = 200 -40x
Critical Points: P'(x) = 0 200-40x = 0 so 200 = 40 x 5 = x
P'(0) = 200>0 Incr

P'(10) = 200 - 400 = -200 <0 Decr

x=5 is a relative max. Since it is the only critical point it is an absolute max.

Conclude: The price should be 400-10(5) =350 dollars
The profit is P(5) = 4000 +200(5) -20(5)^2
=4500 dollars.