4.2 Antiderivatives as Areas

Definite Integrals

Indefinite Integral

If $\int f(x)\,dx = F(x) + C$ (so that $F'(x) = f(x)$)

then

definite integral

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

Example $\int_1^3 x^2\,dx = \frac{1}{3}x^3\Big|_1^3 = \frac{1}{3}3^3 - \frac{1}{3}1^3 = \frac{27-1}{3} = \frac{26}{3}$

Definite Integrals

Properties: $\int_a^a f(x)\,dx = 0$ $(F(a)-F(a) = 0)$

If a<b and If $f(x) \geq 0$ then $\int_a^b f(x)\,dx \geq 0$ $(F'(x) = f(x) \geq 0$ so $F(b) \geq F(a))$

$F(b) - F(a) \geq 0$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ $(F(b) - F(a) = (F(b)-\cancel{F(c)})+(\cancel{F(c)}-F(a))$

$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ $F(a) - F(b) = -(F(b) - F(a))$

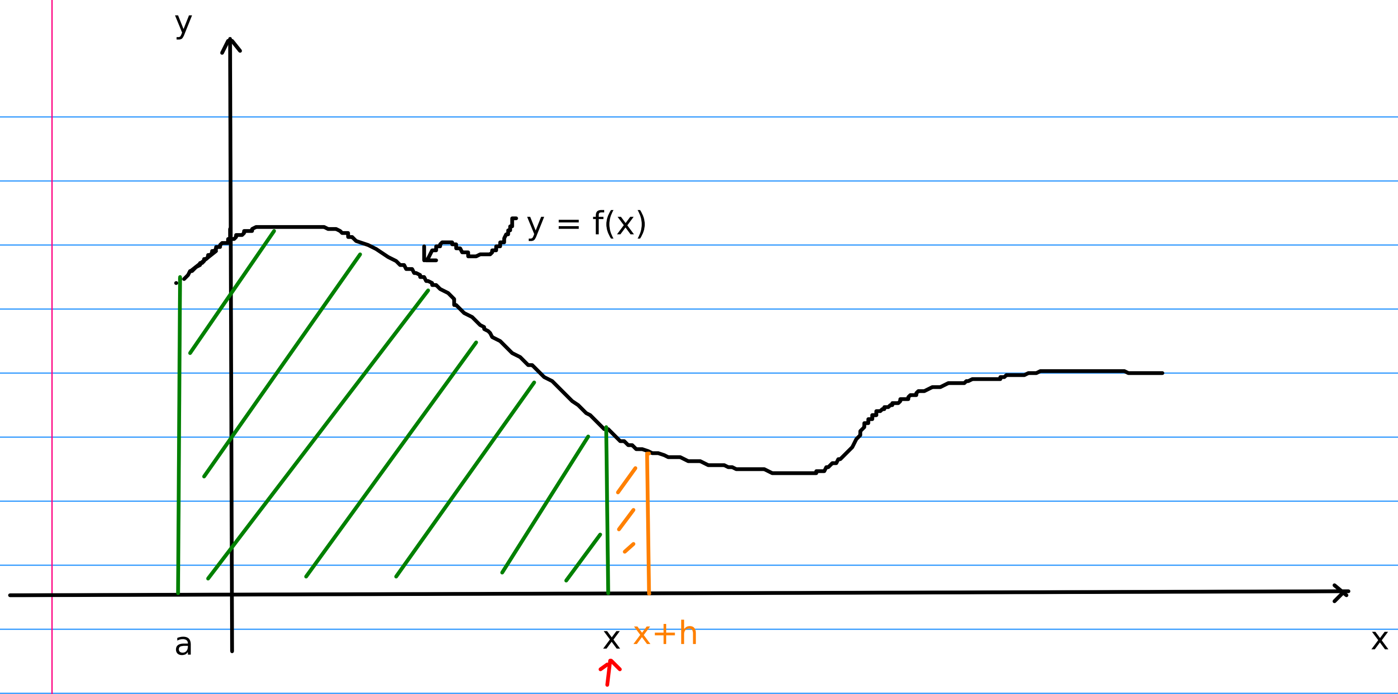
$\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$ $(kF)' = k F'$

$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$ $F(b)+G(b) -(F(a) +G(a))$

$=F(b) - F(a) + G(b)-G(a)$

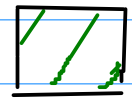
$F' = f$ $G' = g$ $(F+G)' = f+g$

Physical Interpretation of the definite integral as area



Suppose that $f(x) \geq 0$ for $x \geq a$ and f(x) is continuous.

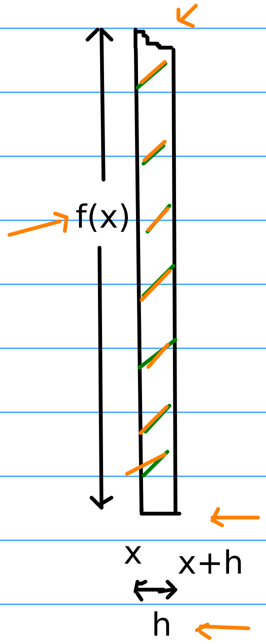
Define A(x) to be the area under the graph of y=f(x) but above the x-axis between a and x.

A(x) = Area of 

We claim that A(x) is a differentiable function of x

Consider $\lim_{h \to 0} \frac{A(x+h)-A(x)}{h}$

What is A(x+h) - A(x) ? It is the area of a region that looks like (after magnification)



A(x+h)-A(x) is the area of a region that looks like a rectangle (almost) of width h and height f(x)

$$A(x+h) - A(x) \simeq f(x)h$$

and the approximation gets better the smaller h is.

$$\lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = f(x)$$

Therefore $A'(x) = f(x)$ or A is an antiderivative of f(x). Since $A(a) = 0$

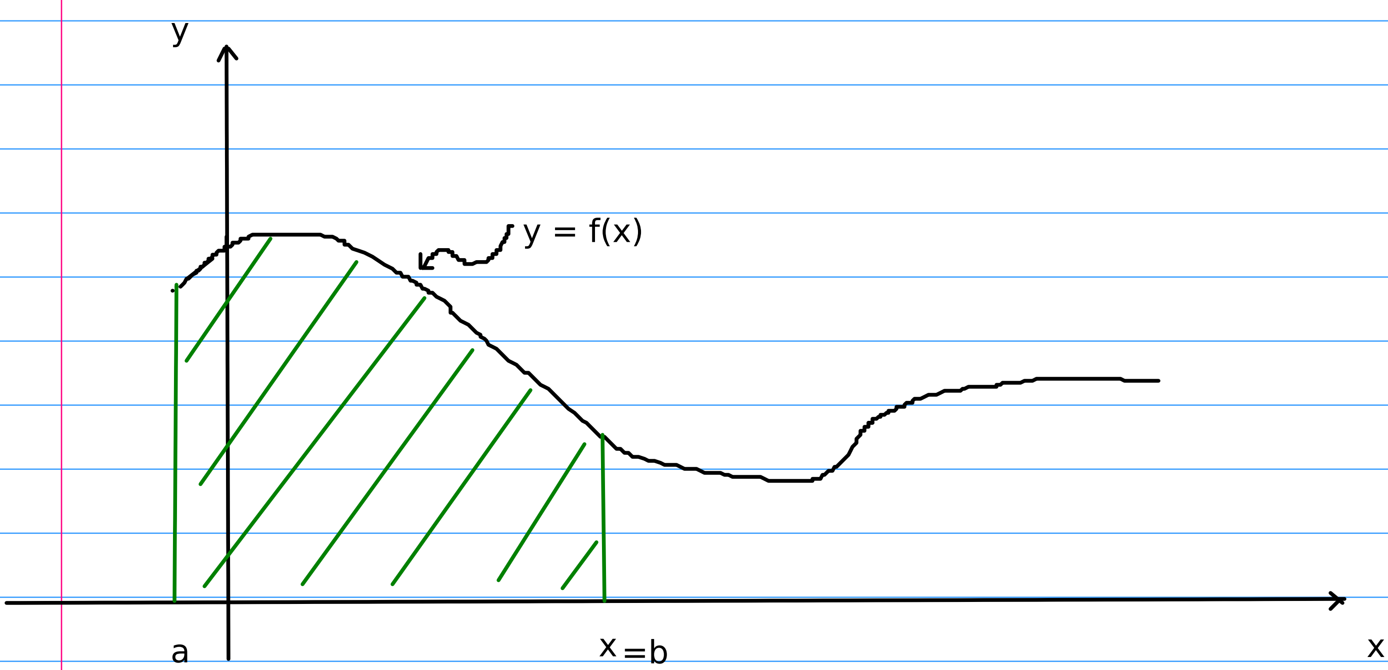
$$\boxed{A(x) = F(x) - F(a) = \int_a^x f(t)\,dt}$$

Conclusion: If $f(x) \geq 0$ and f(x) is continuous then

$$\int_a^b f(x)\,dx = \text{Area under the graph } y = f(x) \; a \leq x \leq b$$

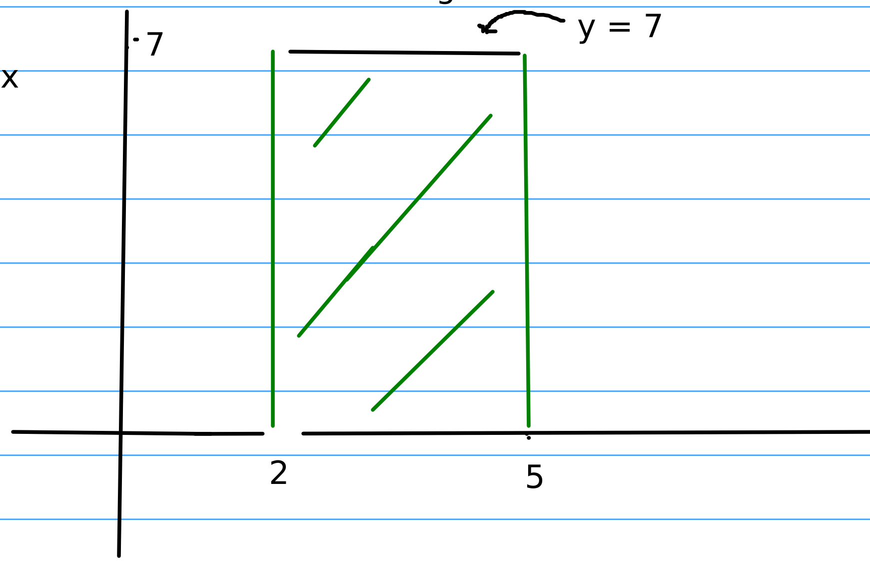
but above the x - axis.

= Area of



Example: Use the interpretation of the definite integral as area to evaluate

$$\int_2^5 7 \, dx$$



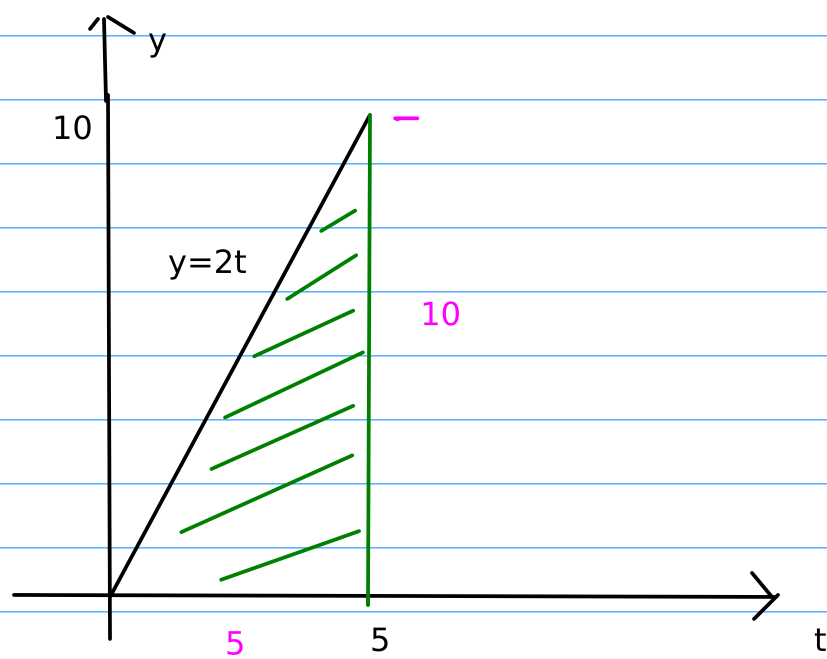
= length x width

= 7 x 3 = 21

$$\int_2^5 7 \, dx = 7x \Big|_2^5 = 7(5) - 7(2) = 35-14 = 21$$

Example: Use the interpretation of the definite integral as area to evaluate

$$\int_0^5 2t \, dt$$



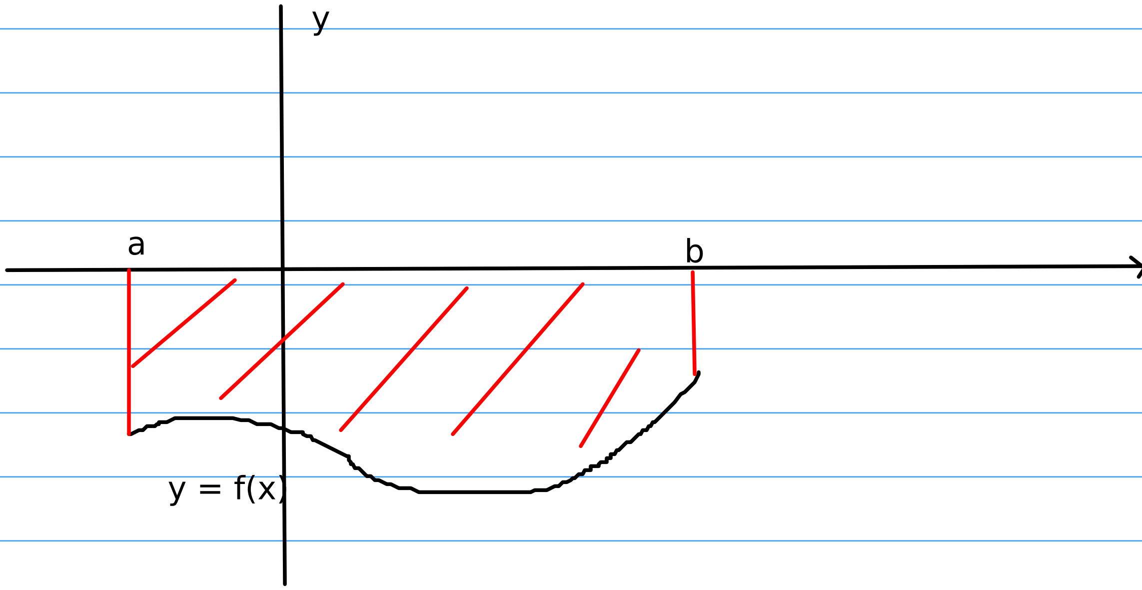
= bh/2

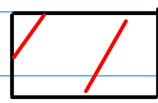
= 10(5) /2 = 25

$$\int_0^5 2t \, dt = t^2 \Big|_0^5 = 5^2 - 0^2 = 25$$

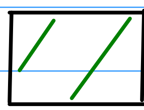
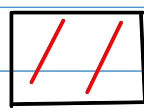
~~Riemann Sums~~ and definition of the Riemann (definite) integral

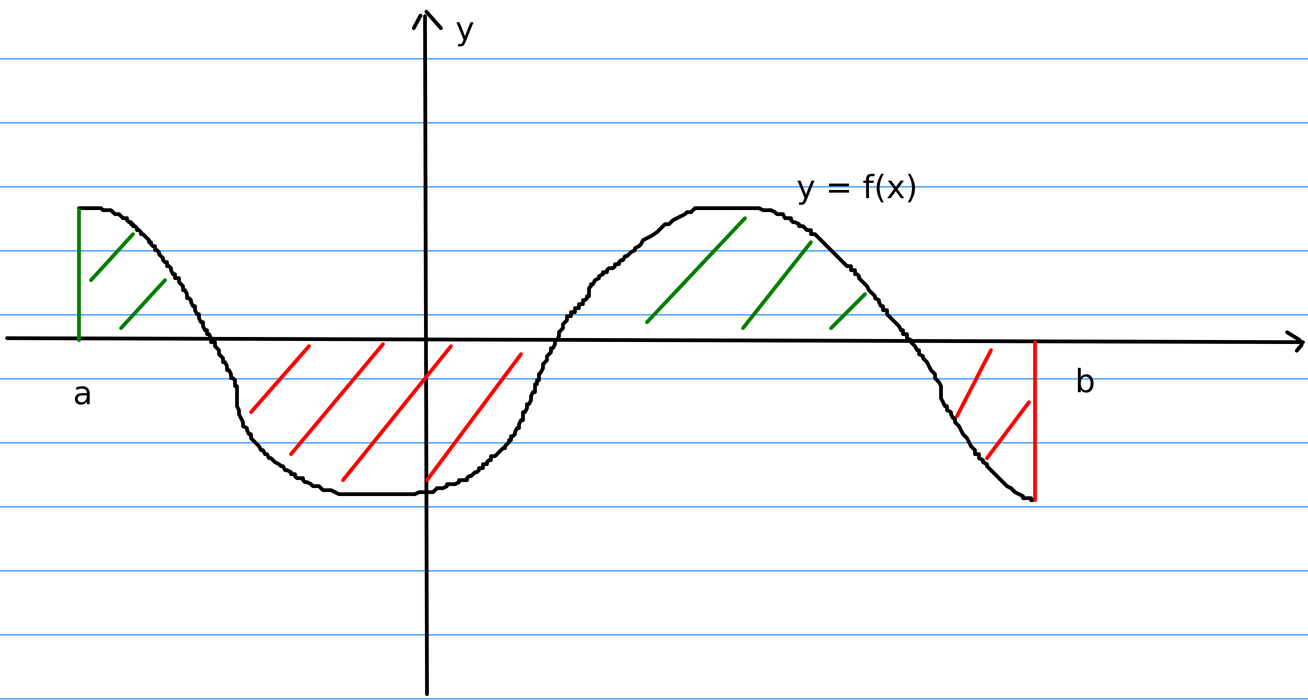
What happens if $f(x) < 0$, $a < x < b$



$\int_a^b f(x)\, dx$ = (-1) Area 

(because $\int_a^b f(x)\, dx = -\int_a^b -f(x)\, dx$ and $-f(x)>0$)

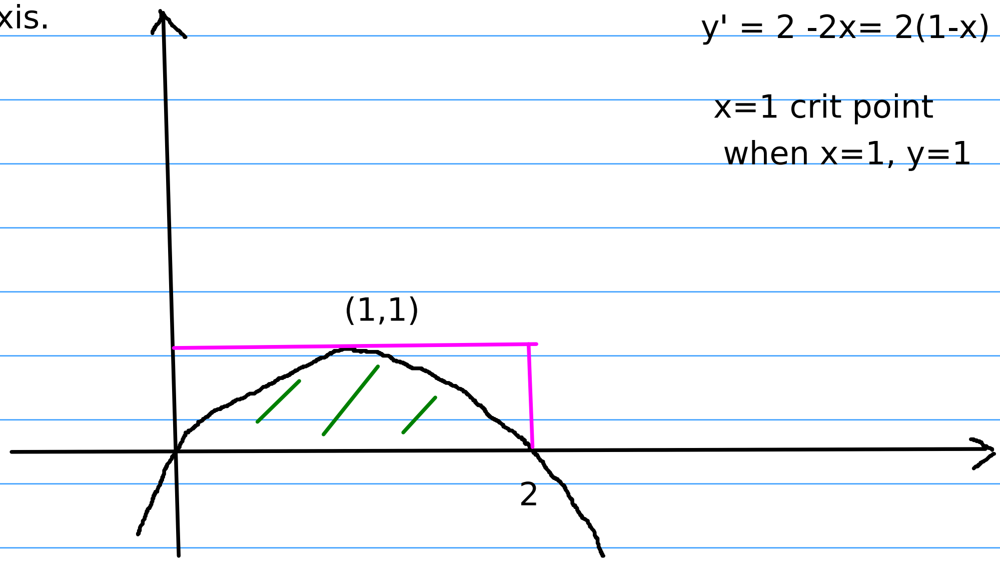
In general $\int_a^b f(x)\, dx$ = Area  - Area 



Example   Evaluate the area of the region below the graph of $y = 2x-x^2$ and above the x-axis.

$y' = 2 -2x= 2(1-x)$

$x=1$ crit point
when $x=1$, $y=1$



$$\text{Area} = \int_0^2 2x - x^2 \, dx = \left[x^2 - \frac{1}{3}x^3 \right]_0^2 = \left[2^2 - \frac{1}{3} 2^3\right] - 0$$

$$= 4 - \frac{8}{3} = \frac{4}{3}$$

Remark: Every continuous function $f(t)$, $a \leq t < b$ has an antiderivative

$$F(x) = \int_a^x f(t)\, dt \quad (= A(x))$$

that is $F'(x) = f(x)$. (This is a consequence of the fact that the area between the graph of f(t) and the x-axis is a well defined. This fact uses the ``Riemann integral.'')

$F'(x) =$

Monday July 29

4.3 Example: Kitchen Design sells countertops. The marginal cost of finishing and installing x centimeters of countertop of a particular style is

$$C'(x) = 12x^{-1/2} \qquad \text{(in dollars per cm)}$$

This month there are already orders for 900 cm of this style of countertop. How much would a further order of 700 cm add to the cost?

$$\text{Extra Cost} = \int_{900}^{1600} 12\, x^{-1/2}\, dx \qquad (= C(1600) - C(900))$$

$$= 12 \int_{900}^{1600} x^{-1/2}\, dx$$

$$= \frac{12}{1/2}\, x^{1/2} \Big|_{900}^{1600}$$

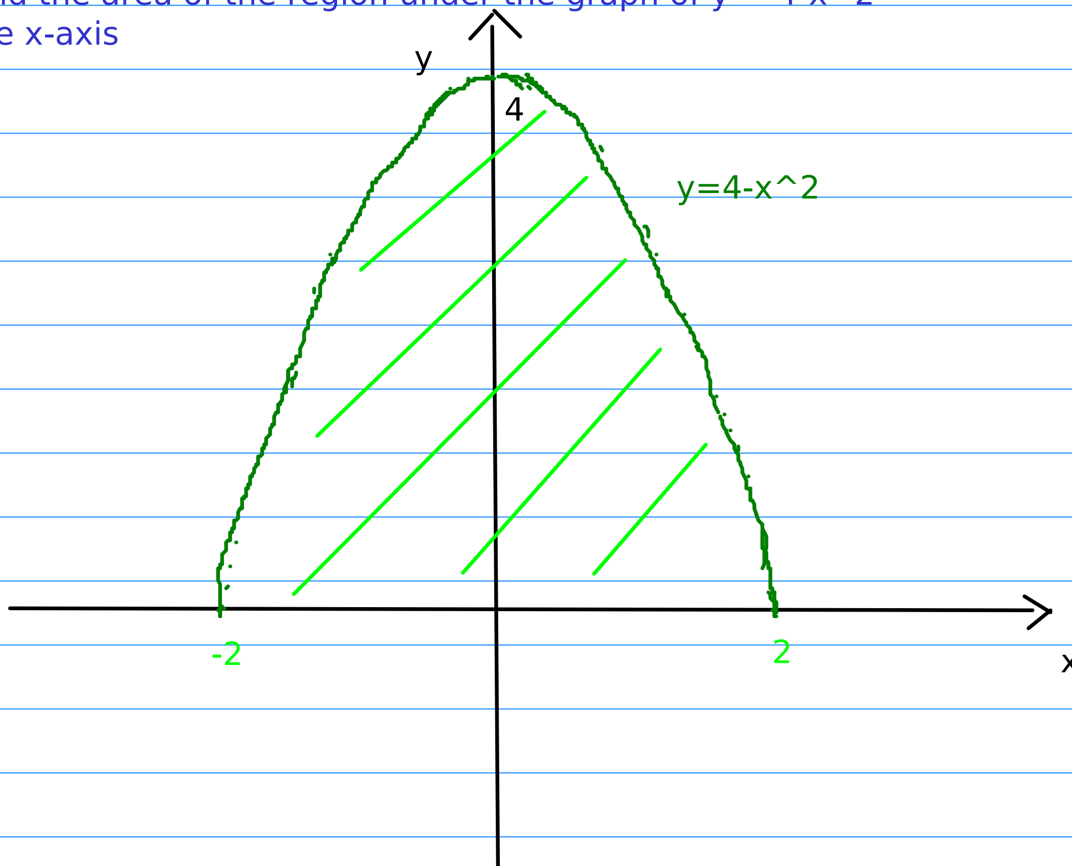
$$= 24 \; [\, 1600^{1/2} - 900^{1/2} \,]$$

$$= 24\, [40 - 30] = \underline{240}$$

Marginal Cost, marginal revenue and marginal profit.

4.4 Properties of Definite Integrals

Example: Find the area of the region under the graph of $y = 4-x^2$ but above the x-axis



Where does the curve $y = 4 - x^2$ meet the x-axis?

$4-x^2 = ??$

$4-x^2 = 0$

Label the picture $(2-x)(2+x) = 0$: $x=2$ or $x = -2$

$$\text{Area} = A = \int_{-2}^{2} 4-x^2 \, dx = 4x - \frac{1}{3}x^3 \Big|_{-2}^{2} = 4(2) - \frac{1}{3}2^3 - \left[4(-2) - \frac{1}{3}(-2)^3\right]$$

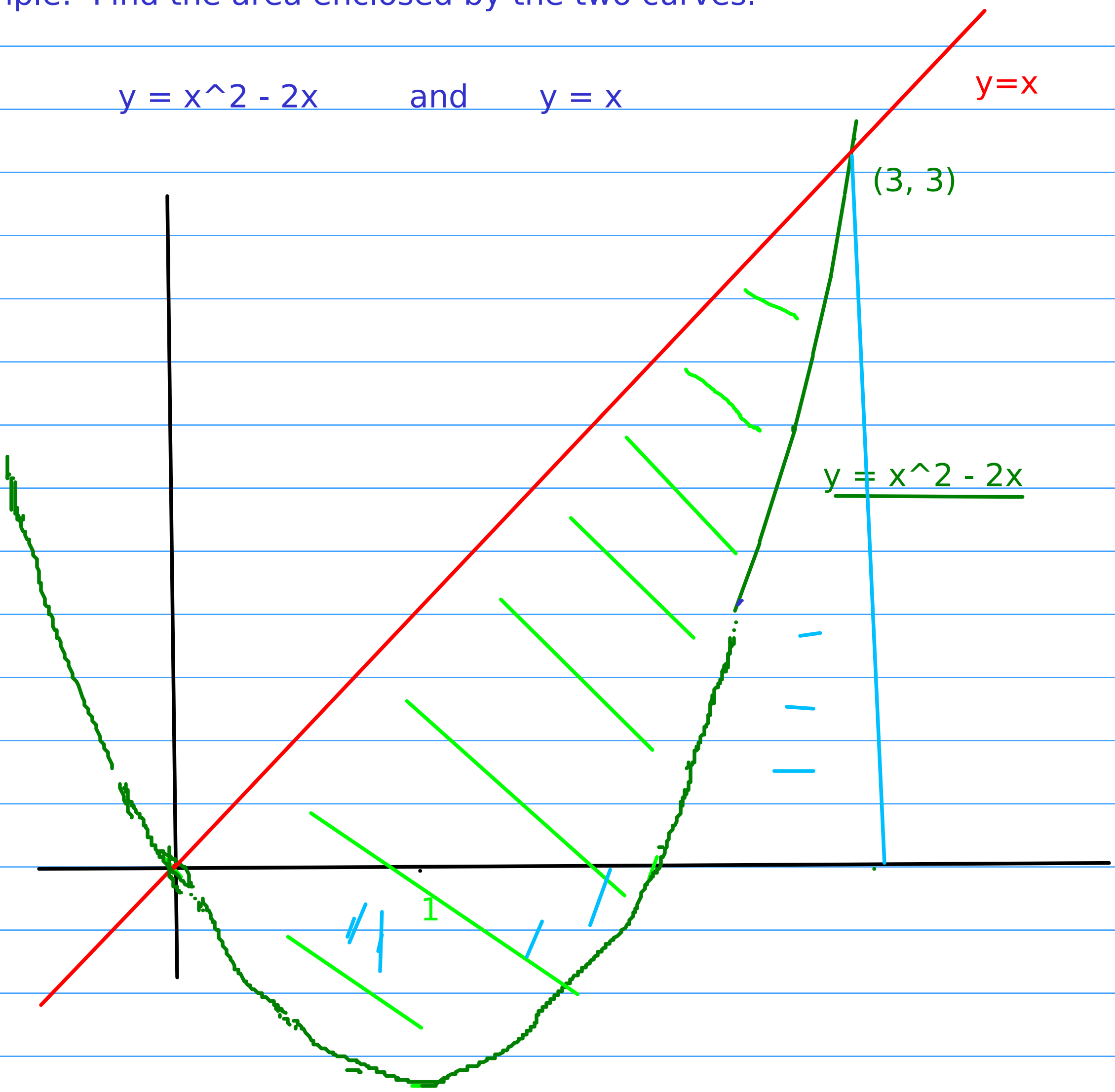
$$= 8 - \frac{8}{3} - \left[-8 + \frac{8}{3}\right]$$

$$= \frac{24-8}{3} + \left[8 - \frac{8}{3}\right]$$

$$= 32/3$$

Example: Find the area enclosed by the two curves.

$y = x^2 - 2x$ and $y = x$



Where do the curves intersect?

$y = x^2 - 2x$ and $y = x$

$x^2 - 2x = x$

$x^2 - 3x = 0$ $\quad x(x-3) = 0$

$x=0$ or $x = 3$

$$\text{Area} = \int_0^3 x - (x^2-2x)\,dx = \int_0^3 3x - x^2\,dx$$

Equation of the upper curve minus the equation of the lower curve

$$= 3\left((1/2)\,x^2\right) - \frac{1}{3}x^3 \Big|_0^3$$

$$= 3/2\,(9) - 9 = 9/2$$

4.5 Integration by Substitution

Recall the

Chain Rule $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

which we srite in the form (u =g(x))

$$\frac{d}{dx} f(u) = f'(u)\, u'$$

or (differential notation) $df(u) = f'(u)du = f'(u)\, u'\, dx$

We reverse this

Example:Determine

$$\int (2x+4)^7\, dx =$$

$u = 2x+4 \qquad \frac{du}{dx} = 2 \quad$ or $\quad du = 2\, dx$ $\qquad$ (Aside $df = f'(x)\, dx$)

$(1/2)\, du = dx$

$$\int (2x+4)^7\, dx = \int u^7\, (1/2)\, du = (1/2) \int u^7 du = (1/2)\ \frac{1}{8} u^8 + C$$

$$= \frac{1}{16} (2x+4)^8 + C$$

Check by differentiation

$$\frac{d}{dx} \frac{1}{16} (2x+4)^8 = \frac{1}{16}\ 8(2x+4)^7\ 2 = \frac{\cancel{8(2)}}{\cancel{16}} (2x+4)^7 \checkmark$$

Example

$$\int x(x^2+3)^9\, dx = \frac{1}{2} \int u^9\, du = \frac{1}{2}\ \frac{1}{10}\ u^{10} + C$$

$u = x^2 + 9 \qquad du = 2x\, dx$

$(1/2)\, du = x\, dx$

$$= \frac{1}{20} (x^2+3)^{10} + C$$

Check by differentiation

$$\frac{d}{dx} \frac{1}{20} (x^2+3)^{10} = \frac{1}{\cancel{20}_{\cancel{2}}}\ \cancel{10}\ (x^2+3)^9\ \cancel{2}x$$

$$= x\, (x^2+3)^9 \checkmark$$

Example $\int_0^1 x(x^2+3)^9\,dx$

On the one hand this must be $\frac{1}{20}(x^2+3)^{10}\Big|_0^1 = \frac{1}{20}(1^2+3)^{10} - \frac{3^{10}}{20}$

$= \frac{1}{20}[4^{10} - 3^{10}]$

OR

Substitute $u = x^2 + 3$ $du = 2x\,dx$ $x=0$ implies $u = 0^2 + 3 = 3$

$x=1$ implies $u = 1^2 + 3 = 4$

$\int_0^1 x(x^2+3)^9\,dx = \frac{1}{2}\int_3^4 u^9\,du = \frac{1}{2}\,\frac{1}{10}\,u^{10}\Big|_3^4$

$= \frac{1}{20}[4^{10} - 3^{10}]$

Example $\int \sqrt{x^2+x+3}\,(2x+1)\,dx$

Substitute $u = x^2 + x + 3$ $\quad du = (2x+1)\,dx$

$$\int \sqrt{x^2+x+3}\,\underbrace{(2x+1)\,dx}_{du} = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}(x^2+x+3)^{3/2} + C$$

Check

$$\frac{d}{dx}\frac{2}{3}(x^2+x+3)^{3/2} = \frac{2}{3}\,\frac{3}{2}(x^2+x+3)^{1/2}(2x+1)$$

$$=(x^2+x+3)^{1/2}(2x+1)$$

Example Determine $\int t\,e^{-t^2}\,dt$

Substitute $u = -t^2$ $\quad du = -2t\,dt$ $\quad -(1/2)\,du = t\,dt$

$$\int t\,e^{-t^2}\,dt = -(1/2)\int e^u\,du = -(1/2)\,e^u + C$$

$$= -(1/2)\,e^{-t^2} + C$$

$$\frac{d}{dt}\,-(1/2)\,e^{-t^2} = -(1/2)\,e^{-t^2}(-2t) = t\,e^{-t^2} \checkmark$$

$$\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \ln x + C$$