e^x $\ln x$

4.1 Antidifferentiation: (Integral Calculus)

Example: Suppose y' = 2x. Then what is y?

y = x^2. Another choice is y = x^2 + 5: y'= 2x

Is y uniquely determined by its derivative? If z' = y' =2x then (z-y)' =z'-y' =2x-2x=0

and so (z-y) is a constant. (Mean ValueTheorem $f(b) - f(a) = \underbrace{f'(c)}_{=0}(b-a)$ a<c<b)

Notation: The general antiderivative of 2x is

$$\int 2x\,dx = x^2 + C$$

integral sign

Examples What is the general antiderivative of

a) $3x^2$ $\int 3x^2\,dx = x^3 + C$ Check $\frac{d}{dx}x^3 = 3x^2$

b) x^4 $\int x^4\,dx = \frac{1}{5}x^5 + C$ Check $\frac{d}{dx}\frac{1}{5}x^5 = \frac{1}{\cancel{5}}\cancel{5}x^4 = x^4$

In general $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ $n \neq -1$

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln x + C$$

Example: $\int x^{11}\,dx = \frac{1}{12}x^{12} + C$

Example $\int x^5 + \frac{2}{x}\,dx = \int x^5\,dx + 2\int \frac{1}{x}\,dx$

$= \frac{1}{6}x^6 + 2\ln x + C$

Check $\frac{d}{dx}\left(\frac{1}{6}x^6 + 2\ln x\right) = \frac{1}{\cancel{6}}\cancel{6}x^5 + 2\frac{1}{x} = x^5 + \frac{2}{x}$ ✓

Example: $\int \sqrt{x}\, dx = \int x^{1/2} dx = \frac{1}{3/2} x^{3/2} + C = \frac{2}{3} x^{3/2} + C$

Example: $\int \frac{1}{x^{1/3}} dx = \int x^{-1/3} dx = \frac{1}{2/3} x^{2/3} + C = \frac{3}{2} x^{2/3} + C$

Example: $\int e^x\, dx = e^x + C$ $\qquad \frac{d}{dx} e^x = e^x \longleftarrow$

Example: Evaluate $\int e^{5x}\, dx = \underline{\frac{1}{5} e^{5x} + C}$

We want something which has derivative e^{5x}:

chain rule

$\underline{\frac{d}{dx} e^{5x} = e^{5x}\, 5}$ Therefore e^{5x} is an antiderivative of $5\, e^{5x}$

Therefore $\underline{\frac{d}{dx} \frac{1}{5} e^{5x} = \frac{1}{\cancel{5}} \cancel{5}\, e^{5x} = e^{5x}}$

In general $\int e^{ax}\, dx = \frac{1}{a} e^{ax} + C$

$\frac{d}{dx} e^{ax} = a\ e^{ax}$

Example: Determine $\int e^{0.02\, t}\, dt = \frac{1}{0.02} e^{0.02\, t} + C$

$= 50\, e^{0.02t} + C$

Initial conditions and the Constant of Integration.

Example: Suppose $f'(x) = \sqrt{x} + 3x^2$ and $\underline{f(1) = 5}$. What is f?

Solution: $f(x) = \int f'(x)\,dx = \int x^{1/2} + 3x^2\,dx$

$$= \frac{1}{3/2} x^{3/2} + x^3 + C$$

$$= \frac{2}{3} x^{3/2} + x^3 + C$$

Now evaluate at x = 1

$$5 = f(1) = (2/3)\,1^{3/2} + 1^3 + C$$

$C = 5 - 5/3 = 10/3$

$$f(x) = (2/3)\,x^{3/2} + x^3 + \frac{10}{3}$$

Example: The velocity of a projectile is
v(t) = 64 - 32 t where t is time in seconds after the projectile is thrown at 64 ft/second The the projectile is released at 12 feet above ground level then what is its height?

$v(t) = h'(t)$ where h is the height of the projectile.

$$h(t) = \int 64 - 32t \; dt = 64t - 32\left(\frac{1}{2}t^2\right) + C = 64t - 16t^2 + C$$

But h(0) = 12. 12 = c and so $12 = h(0) = \cancel{64(0) - 16(0)^2} + C$

h(t) = 64 t - 16 t^2 + 12

Example: Suppose the marginal cost of producing a soft drink is

$$C'(x) = 0.0003x \quad + \; .02\,x \quad + \; 1$$

in dollars per gallon.

$$C(x) = \int C'(x)\,dx$$

Suppose that 100 gallons have already been produced. What is the cost of producing the next 200 gallons

$$C(300)\text{-}C(100) = \quad 0.0003\,x \quad +0.02\,x \; + 1 \;\; dx$$

4.2 Antiderivatives as Areas

Definite Integrals

Indefinite Integral

If $\int f(x)\,dx = F(x) + C$ (so that $F'(x) = f(x)$)

then definite integral

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

Example $\int_1^3 x^2\,dx = \frac{1}{3}x^3\Big|_1^3 = \frac{1}{3}3^3 - \frac{1}{3}1^3 = \frac{27-1}{3} = \frac{26}{3}$

Definite Integrals

Properties: $\int_a^a f(x)\,dx = 0$ $(F(a)-F(a) = 0)$

If $a<b$ and If $f(x) \geq 0$ then $\int_a^b f(x)\,dx \geq 0$ $(F'(x) = f(x) \geq 0$ so $F(b) \geq F(a))$

$F(b) - F(a) \geq 0$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ $(F(b) - F(a) = (F(b)\cancel{-F(c)})+(\cancel{F(c)}-F(a))$

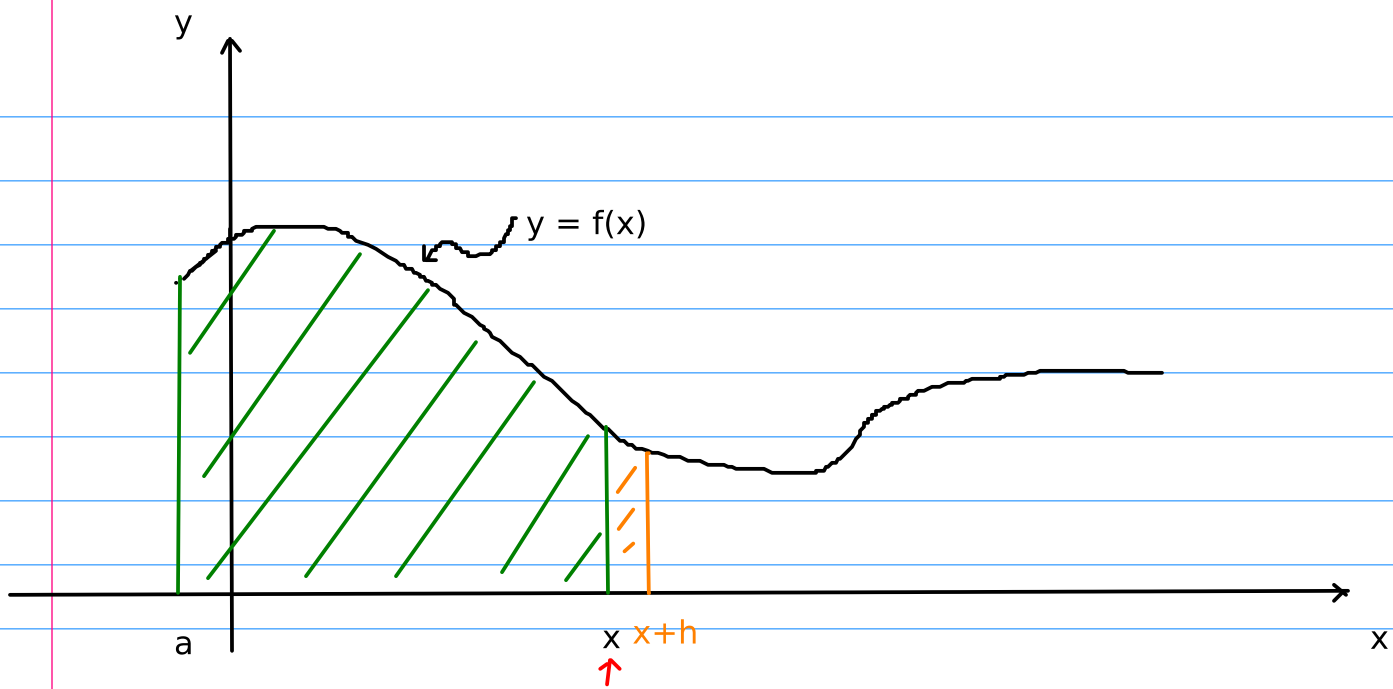
$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ $F(a) - F(b) = -(F(b) - F(a))$

$\int_a^b k\, f(x)\,dx = k\int_a^b f(x)\,dx$ $(kF)' = k\,F'$

$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$ $F(b)+G(b) -(F(a) +G(a))$
$=F(b) - F(a) + G(b)-G(a)$

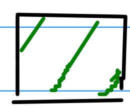
$F' = f$ $G' = g$ $(F+G)' = f+g$

Physical Interpretation of the definite integral as area



Suppose that $f(x) \geq 0$ for $x \geq a$ and $f(x)$ is continuous.

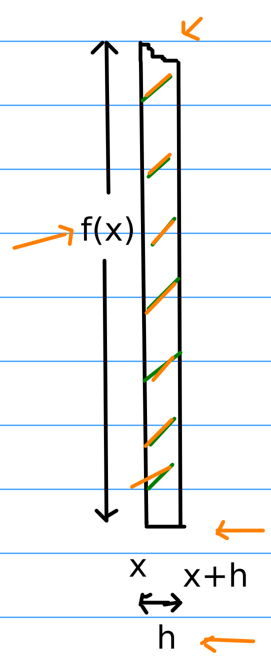
Define $A(x)$ to be the area under the graph of $y=f(x)$ but above the x-axis between a and x.

$A(x)$ = Area of 

We claim that $A(x)$ is a differentiable function of x

Consider $\lim_{h \to 0} \frac{A(x+h)-A(x)}{h}$

What is $A(x+h) - A(x)$? It is the area of a region that looks like (after magnification)



A(x+h)-A(x) is the area of a region that looks like a rectangle (almost) of width h and height f(x)

$$A(x+h) - A(x) \simeq f(x)h$$

and the approximation gets better the smaller h is.

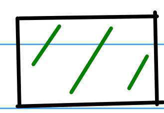
$$\lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = f(x)$$

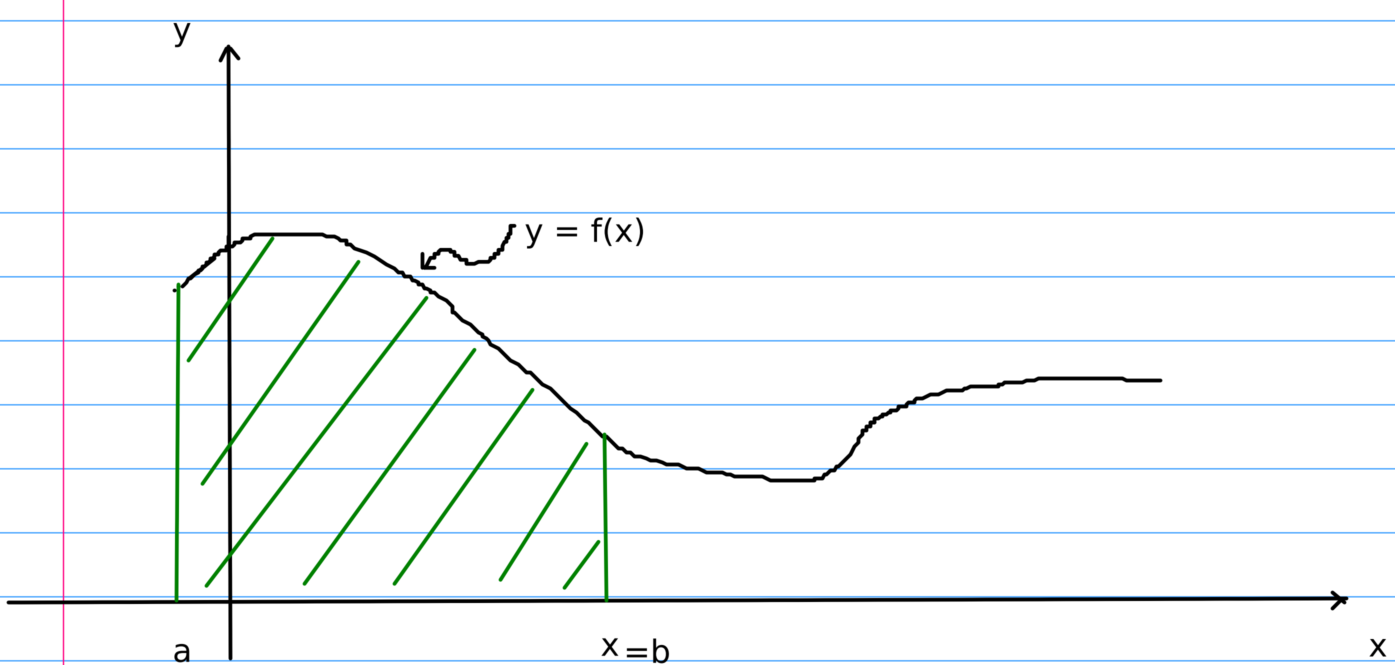
Therefore $A'(x) = f(x)$ or A is an antiderivative of f(x). Since $A(a) = 0$

$$\boxed{A(x) = F(x) - F(a) = \int_a^x f(t)\,dt}$$

Conclusion: If $f(x) \geq 0$ and f(x) is continuous then

$$\int_a^b f(x)\,dx =$$ Area under the graph $y = f(x)$ $a \leq x \leq b$ but above the x - axis.

= Area of 

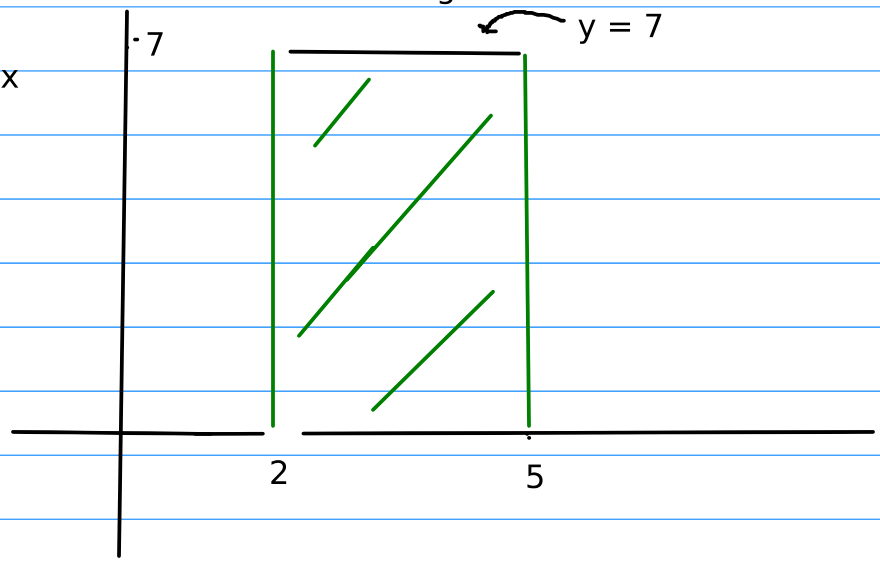


Example: Use the interpretation of the definite integral as area to evaluate

$$\int_2^5 7 \, dx$$

$= \text{length x width}$

$= 7 \times 3 = 21$



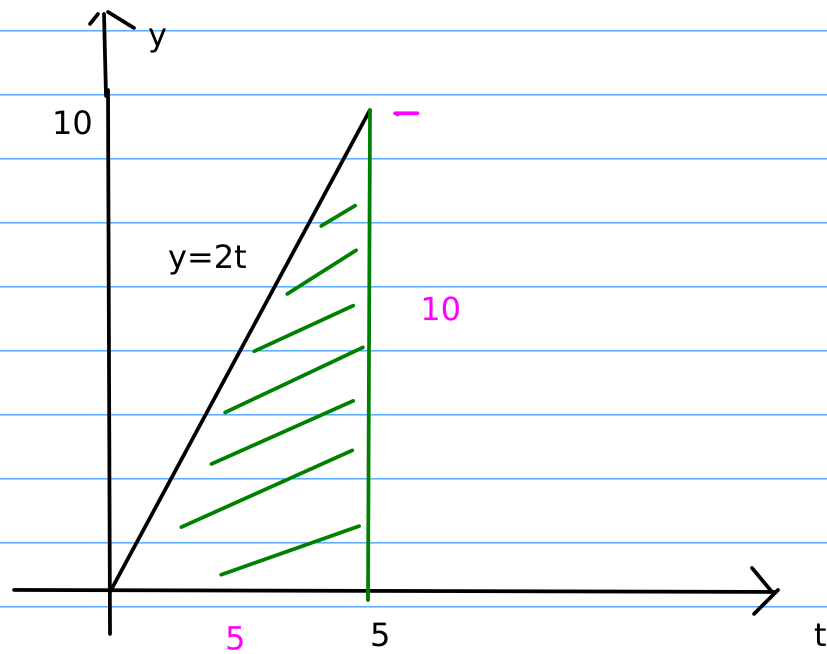
$$\int_2^5 7\,dx = 7x \Big|_2^5 = 7(5) - 7(2) = 35\text{-}14 = 21$$

Example: Use the interpretation of the definite integral as area to evaluate

$$\int_0^5 2t \, dt$$

$= bh/2$

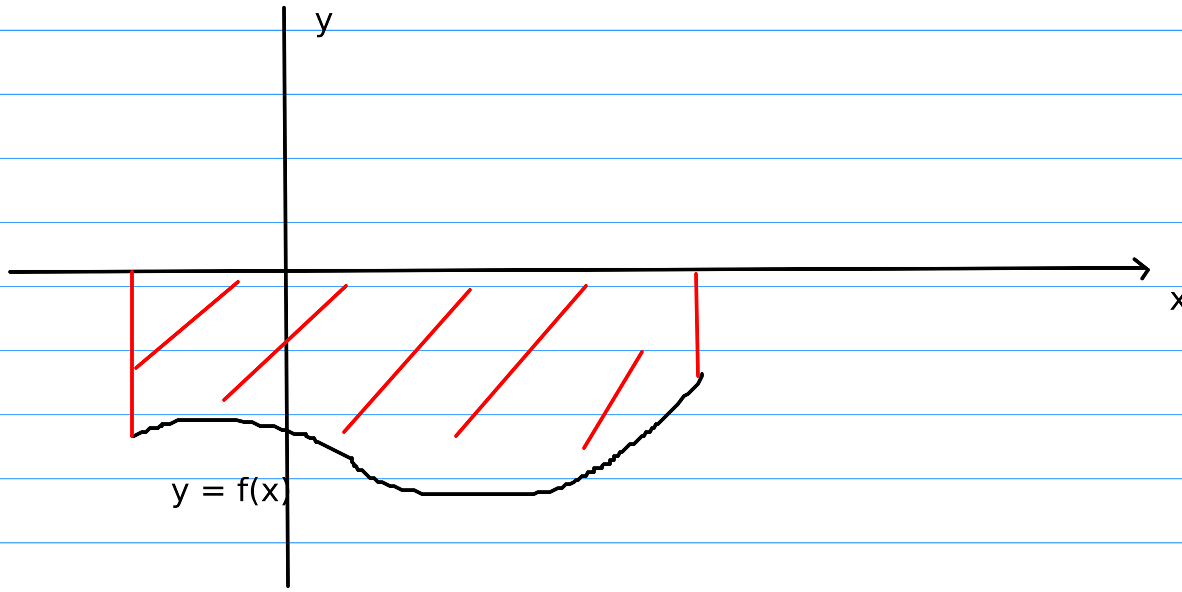
$= 10(5)/2 = 25$

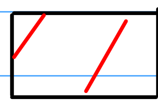


$$\int_0^5 2t\,dt = t^2 \Big|_0^5 = 5^2 - 0^2 = 25$$

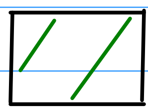
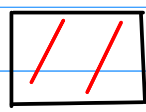
~~Riemann Sums~~ and definition of the Riemann (definite) integral

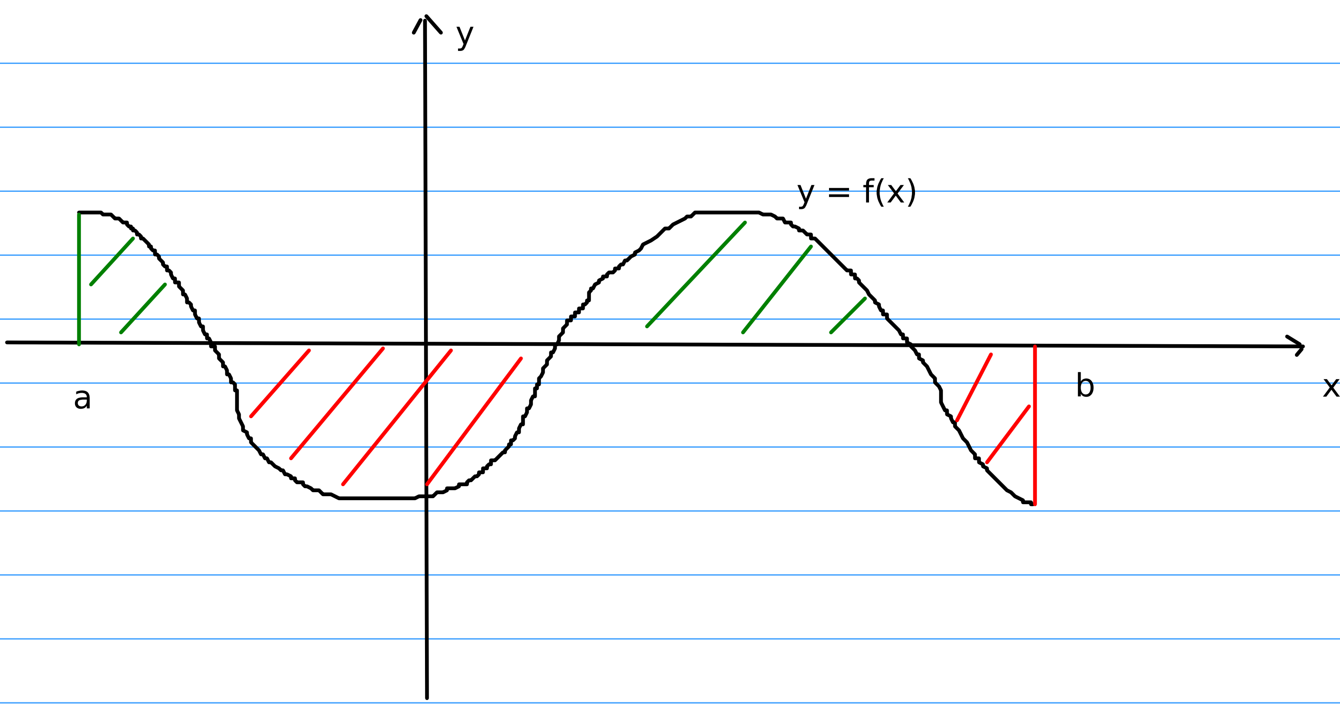
What happens if $f(x) < 0$, $a < x < b$



$\int_a^b f(x)\,dx$ = (-1) Area 

(because $\int_a^b f(x)\,dx = -\int_a^b -f(x)\,dx$ and $-f(x)>0$)

In general $\int_a^b f(x)\,dx$ = Area  - Area 

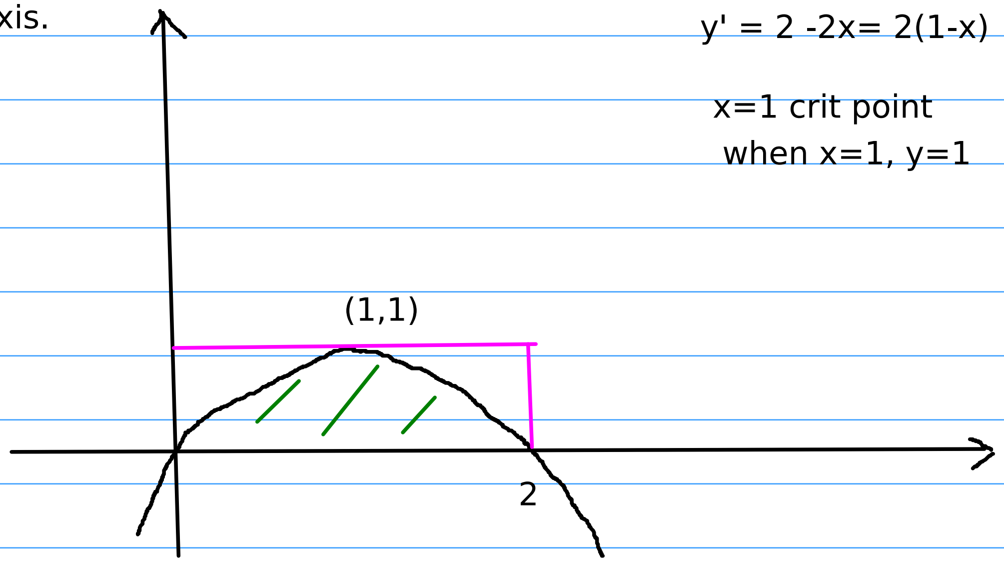


Example Evaluate the area of the region below the graph of $y = 2x-x^2$ and above the x-axis.

$y' = 2 - 2x = 2(1-x)$

$x=1$ crit point

when $x=1$, $y=1$



$$\text{Area} = \int_0^2 2x - x^2 \; dx = \left[x^2 - \frac{1}{3}x^3 \right]_0^2 = \left[2^2 - \frac{1}{3} 2^3 \right] - 0$$

$$= 4 - \frac{8}{3} = \frac{4}{3}$$