3.5 General Logs and Exponentials

a^x and $\log_a$

Recall that $\frac{d}{dx} e^x = \underline{e^x}$

$a = e^{(\ln a)}$

So $a^x = e^{(\ln a\, x)}$

Example $2^x = e^{(\ln 2\, x)} \approx e^{(0.693\, x)}$

In general $a^x = e^{(\ln a)\, x}$

and a and (ln a) are just constants. Therefore

$$\frac{d}{dx} a^x = (\ln a)\, a^x$$

Example $\underline{\frac{d}{dx} 10^x = (\ln 10)\, 10^x} \approx (2.30) 10^x$

Example $\frac{d}{dx} 5^{2x+3} = (\ln 5)\, 5^{2x+3} (\ 2\ )$ by the chain rule (fill in the blank)

Example $\frac{d}{dx} 3^{x^5+2x} = (\ln 3)\, 3^{x^5+2x} (5x^4 + 2\ )$ by the chain rule (fill in the blank)

General Logarithms

Recall $\log_a x = \dfrac{\ln x}{\ln a}$

(because $\dfrac{\ln a^x}{\ln a} = \dfrac{x \ln a}{\ln a} = x$ and of course $\log_a a^x = x$)

Therefore

$$\frac{d}{dx} \log_a x = \frac{d}{dx} \frac{\ln x}{\ln a} = \frac{1}{x \ln a}$$

Example: $\dfrac{d}{dx} \log_3 x = \dfrac{1}{x \ln 3}$

Example $\dfrac{d}{dx} \log_{10} (x^2 + 3) = \dfrac{1}{(x^2 + 3) \ln 10} (2x)$

Example Differentiate $y = \log_2 x^4$

Simplify $y = 4 \log_2 x$ so that $y' = \dfrac{4}{x \ln 2}$

Example Differentiate $y = x^2 \log_{10} \sqrt{x+2} = x^2 \log_{10} (x+2)^{1/2}$

Simplify $y = \dfrac{1}{2} [x^2 \log_{10} (x+2)]$

Therefore, by the product rule

$$y' = \frac{1}{2} \left[x^2 \frac{1}{(\ln 10)(x+2)} + 2x \log_{10} (x+2) \right]$$

$$y' = \frac{x^2}{2(\ln 10)(x+2)} + x \log_{10} (x+2)$$

e^x $\qquad$ $\ln x$

4.1 Antidifferentiation: (Integral Calculus)

Example: Suppose y' = 2x. Then what is y?

y = x^2. Another choice is y = x^2 + 5: y'= 2x

Is y uniquely determined by its derivative? If z' = y' =2x then (z-y)' =z'-y' =2x-2x=0

and so (z-y) is a constant. (Mean ValueTheorem $f(b) - f(a) = \underbrace{f'(c)}_{=0}(b-a)$ $a<c<b$)

Notation: The general antiderivative of 2x is

$$\int 2x\,\underline{dx} = x^2 + C$$

integral sign

Examples What is the general antiderivative of

a) $3x^2$ $\qquad \int 3x^2\,dx = x^3 + C$ $\qquad$ Check $\frac{d}{dx} x^3 = 3x^2$

b) x^4 $\qquad \int x^4\,dx = \frac{1}{5}x^5 + C$ $\qquad$ Check $\frac{d}{dx}\frac{1}{5}x^5 = \frac{1}{\cancel{5}}\cancel{5}x^4 = x^4$

In general $\qquad \int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \qquad n \neq -1$

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln x + C$$

Example: $\int x^{11}\,dx = \frac{1}{12}x^{12} + C$

Example $\int x^5 + \frac{2}{x}\,dx = \int x^5\,dx + 2\int \frac{1}{x}\,dx$

$$= \frac{1}{6}x^6 + 2\ln x + C$$

Example: $\int \sqrt{x}\, dx = \int x^{1/2} dx = \frac{1}{3/2} x^{3/2} + C = \frac{2}{3} x^{3/2} + C$

Example: $\int \frac{1}{x^{1/3}} dx = \int x^{-1/3} dx = \frac{1}{2/3} x^{2/3} + C = \frac{3}{2} x^{2/3} + C$

Example: $\int e^x \, dx = e^x + C$ $\qquad \frac{d}{dx} e^x = e^x$

Example: Evaluate $\int e^{5x} \, dx = \frac{1}{5} e^{5x} + C$

We want something which has derivative e^{5x} :

$\frac{d}{dx} e^{5x} = e^{5x}\, 5$ Therefore e^{5x} is an antiderivative of $5e^{5x}$

Therefore $\frac{d}{dx} \frac{1}{5} e^{5x} = \frac{1}{\cancel{5}} \cancel{5} e^{5x} = e^{5x}$

In general $\underline{\underline{\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C}}$

$\frac{d}{dx} e^{ax} = a \; e^{ax}$

Example: Determine $\int e^{0.02\,t} \, dt = \frac{1}{0.02} e^{0.02\,t} + C$

$= 50\, e^{0.02t} + C$

4.1 Initial conditions and the Constant of Integration.

Example: Suppose $f'(x) = \sqrt{x} + 3x^2$ and $\underline{f(1) = 5}$. What is f?

Solution: $f(x) = \int f'(x)\,dx = \int \underbrace{x^{1/2} + 3x^2}_{f'(x)}\,dx$ $\sqrt{x} = x^{1/2}$

$= \frac{1}{3/2} x^{3/2} + x^3 + C$

$= \frac{2}{3} x^{3/2} + x^3 + C$

Now evaluate at x = 1

$5 = f(1) = \underbrace{(2/3)\,1^{3/2} + 1^3}_{5/3} + C = 5/3 + C$

$C = 10/3$

$f(x) = (2/3)\,x^{3/2} + x^3 + 10/3$

Example: The velocity of a projectile is
v(t) = 64 - 32 t where t is time in seconds after the projectile is
thrown at 64 ft/second The the projectile is released at 12 feet above
ground level then what is its height?

v(t) = h'(t) where h is the height of the projectile.

$h(t) = \int 64 - 32\,t \;\; dt = 64\,t - 32\,(\frac{1}{2}\,t^2) + C = 64t - 16\,t^2 + C$

But h(0) = 12. 12 = c and so

12 = h(0) = 64 (0) - 16 (0)^2 + C

12 = C

h(t) = 64 t - 16 t ^2 + 12

4.2 Antiderivatives as Areas

Indefinite integral $\int f(x)\,dx = F(x) + C$ where $F' = f$

Definite Integrals:

If $\int f(x)\,dx = F(x) + C$ (so that $F'(x) = f(x)$)

then the definite integral is defined to be

$$\int_a^b f(x)\,dx = F(b) - F(a) \equiv F(x)\Big|_a^b$$

Example $\int_1^3 x^2\,dx = \frac{1}{3}x^3\Big|_1^3 = \frac{1}{3}3^3 - \frac{1}{3}1^3 = \frac{27-1}{3} = \frac{26}{3}$

Example $\int_1^4 \sqrt{x}\,dx = \int_1^4 x^{1/2}\,dx = \frac{2}{3}x^{3/2}\Big|_1^4 = \frac{2}{3}4^{3/2} - \frac{2}{3}1^{3/2}$

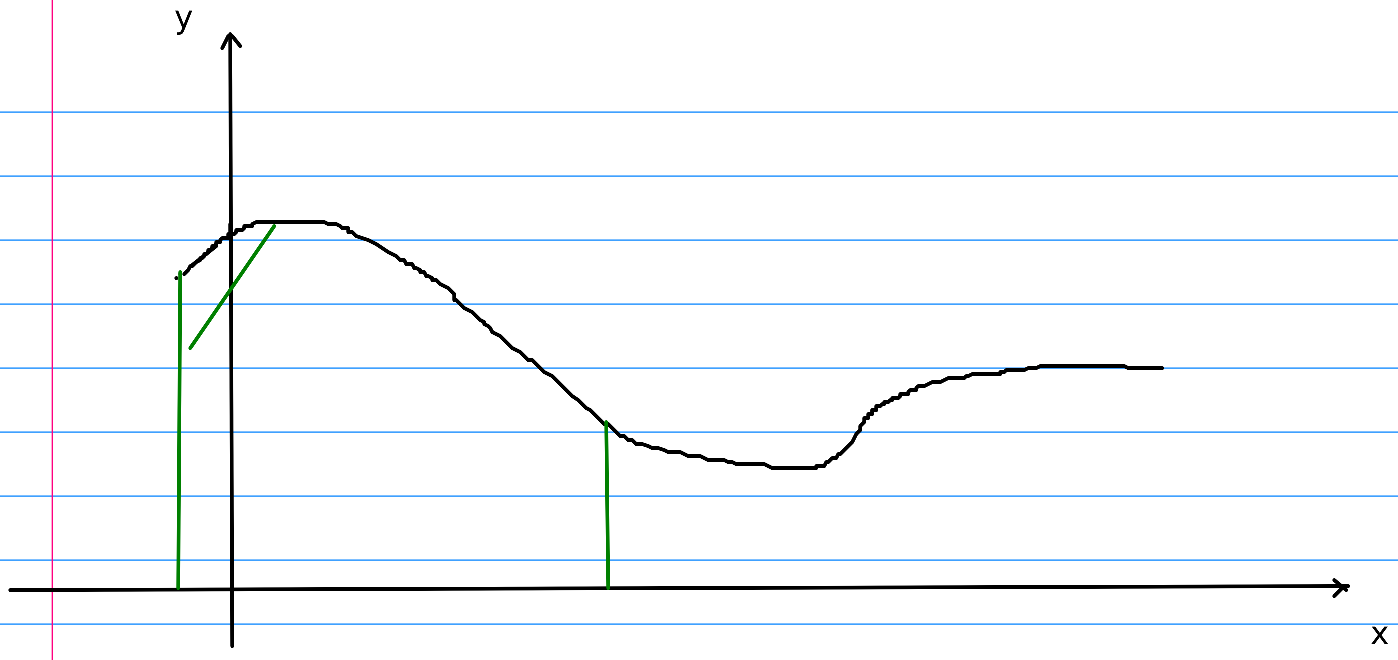
$= \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$

Properties: $\int_a^a f(x)\,dx = 0$ $(F(a)-F(a) = 0)$

If a<b and If $f(x) \geq 0$ then $\int_a^b f(x)\,dx \geq 0$ $(F'(x) = f(x) \geq 0$ so $F(b) \geq F(a))$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ $(F(b) - F(a) = (F(b)-F(c))+(F(c)-F(a)))$

$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$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