5. Find the horizontal asymptotes of $y = \frac{x^2 + 3x+2}{3x^2-3}$

Use limits.

$$\lim_{x \to \infty} y = \lim \frac{x^2}{x^2} \frac{1+3/x+2/x^2}{3-3/x^2} = \lim \frac{1+ 3/x + 2/x^2}{3-3/x^2} = \frac{1+0}{3-0} = 1/3$$

The horizontal asymptote is y = 1/3 at $x = \pm\infty$

$$\lim_{x \to -\infty} y = 1/3$$

6. What is the revenue ?

R(x) = (70-2x)* (300+10x)

What is the cost?

C(x) = 20*(300+10x)

What is the profit?

P(x) = R(x) - C(x) =(70-2x)(300+10x) -20*(300+10x)
= (70-2x-20)(300+10x)

=(50-2x) (300+10x)

= 15000 -100 x -20x^2

P'(x) = -100 -40x
Critical points P'(x) = 0 40x = -100 or x = -5/2 (x<35 also x>-30)
P"(x)=-40<0 Therefore x = -5/2 is a relative max. By max-min principle 2 this is an absolute max. So x = -5/2 and the price should be 70-2*(-5/2)=75
So only 300 +10(-5/2) =275 books will be sold. That will maximize profit.

P(-5/2) = 15000 -100(-5/2)-20(-5/2)^2=15000+250-125 =15125.

3.2 Properties of Logarithms

We write log for $\log = \log_a$

a=2 ; log(2^3*2^4) =log(2^7)= 7

log(2^3) = 3   log(2^4) = 4

1) $\log(xy) = \log(x) + \log(y)$

2) $\log(x/y) = \log(x) - \log(y)$

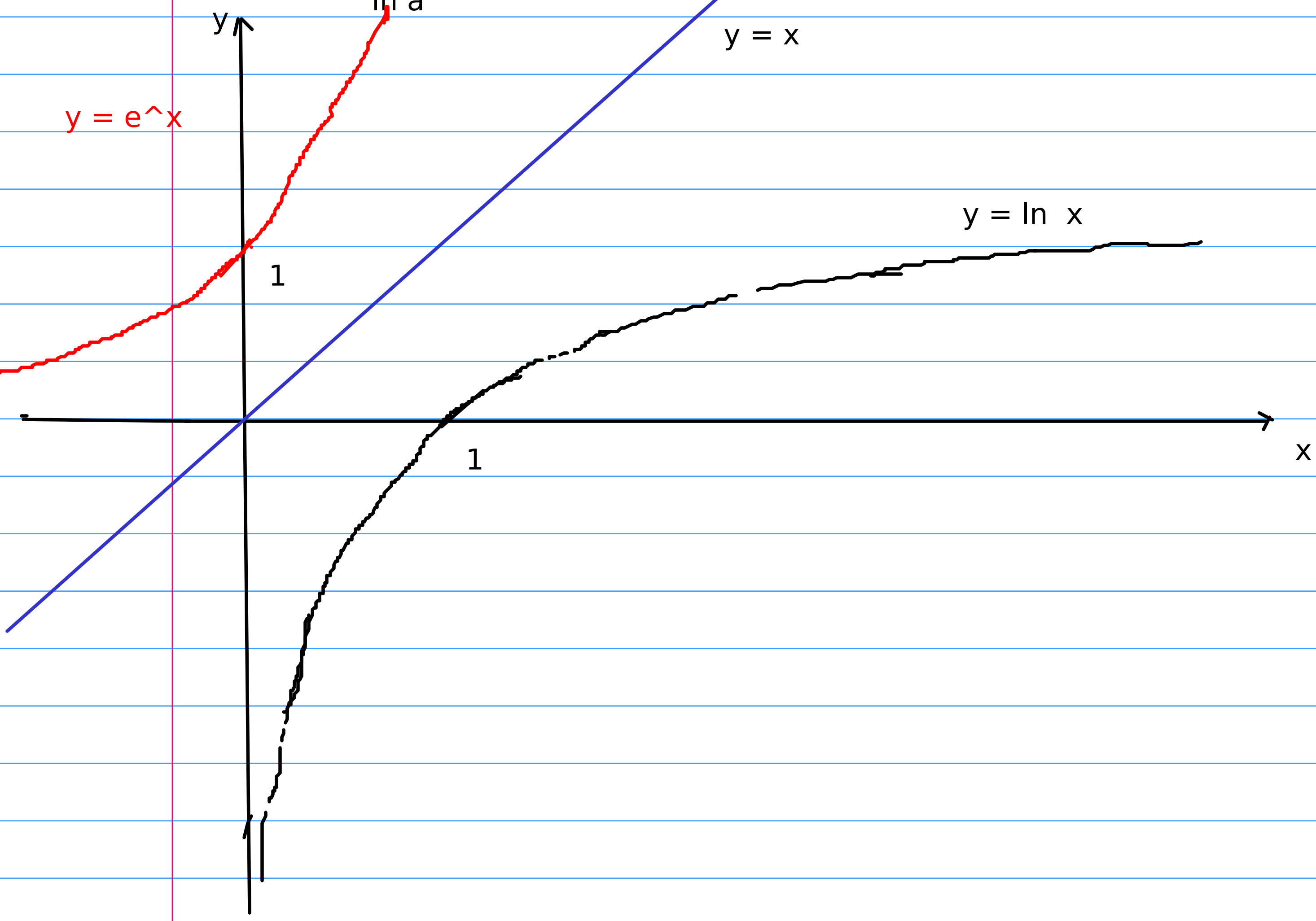
3) $\log(x^k) = k \log(x)$

4) $\log_a a = 1$

5) $\log a^k = k$

6) $\log 1 = 0$

7) $\log x = \dfrac{\ln x}{\ln a}$ (all logs differ by a multiplicative constant.)



8) $\frac{d}{dx} \ln x = \frac{1}{x}$

Rationale: $e^{\ln x} = x$

Differentiate $e^{\ln x} \frac{d}{dx} \ln x = \frac{d}{dx} x = 1$

chain rule

So $x \frac{d}{dx} \ln x = 1$ or

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

Example 1 Differentiate y = ln (x^2+x +2)

Recall the chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Here f(x) = ln x and g(x) = x^2+x+ 2

$y' = \frac{1}{x^2+x+2} (2x+1) = \frac{2x+1}{x^2+x+1}$

Example 2 Differentiate $y = \ln\sqrt{x+1}$

Simplify y = ln (x+1)^(1/2) = $\frac{1}{2} \ln(x+1)$

So $y' = \frac{1}{2} \frac{1}{x+1} = \frac{1}{2(x+1)}$

Example 3 Differentiate $y = \ln[x^2(x+1)(x-3)]$

Simplify $y = \ln x^2 + \ln(x+1) + \ln(x-3) = 2\ln x + \ln(x+1) + \ln(x-3)$

so that $y' = \frac{2}{x} + \frac{1}{x+1} + \frac{1}{x-3}$

Example. Differentiate $y = [\ln x]^5$

Does y simplify? $y' = 5[\ln x]^4 (1/x) = \frac{5[\ln x]^4}{x}$

Monday, July 15

3.3 Applications: Uninhibited and Limited Growth Models

We consider quantities y in our world that grow at a rate proportional to themselves

$$y' = ky$$

where k is the constant of proportionality and might be a positive or negative real number.

$y = Ce^{(kx)}$ is an example of such y for any C. All y are of this form.

(x is time)

$y = 1000e^{(2x)}$ $y' = 1000e^{(2x)} \cdot 2 = 2000e^{(2x)}$

$= 2y$

Examples: 1) Population. The number y of bacteria in a culture grows at a rate proportional to y. It is assumed that the food availability is constant and the death rate is proportional to the population.

2) Radioactive decay. Strontium 90 is a radioactive element and, as such, each atom has a certain probability of decaying into some-thing other than strontium 90 in a given time period. If y is the number of Strontium 90 atoms then $y'=ky$ where $k<0$. The number of strontium 90 atoms is decreasing.

3) A portfolio of investments in bonds grows in value at a rate proportional to the values of the account or this is a close approximation

If y = P is the value of the account after t years then $P' = rP$ where r is the nominal interest rate which is compounded continuously or at least very often

$$P = P_0 e^{(rt)}$$

P_0 is the initial investment. t is time in years.

$r=0.005$ (half of one per cent interest.)

If the interest rate is 3% then the value of the is y where

$$y' = (.03)y$$

roughly. (This approximates compound interest when the compounding period is very short: ``continuous compounding.)

Compound Interest: P_0 is deposited to a bank account that earns interest at the (nominal) rate of 3% per year compounded monthly The t years later the account is worth

$$P = P_0 (1 + i/12)^{(12t)}$$

where i = 0.03 is the rate. If the interest were compounded daily (360 days per year)

$$P = P_0 (1+i/360)^{(360t)}$$

If the compounding were n times per year frequent then

$$P = P_0(1+i/n)^{(nt)} \xrightarrow[n \to \infty]{} P_0 e^{(it)}$$

and as n gets very large and 360 is already large: $P \longrightarrow P_0 e^{(it)}$

Example: An investment portfolio earns 4% interest compounded continuously. Initially the portfolio is worth \$15,000. (a) How much is it worth 6 years later? (b) How long will it take to double in value?

$P' = 0.04P$

interest (decimal)

$$P = P_0 e^{rt} = \underbrace{15000}_{\text{initial investment}} e^{0.04t}$$

(a) the value of the account 6 years later is $P(6) = 15000e^{0.04*6}$ $= \underline{15000e^{0.24}} \approx 19{,}068.74$

(b) How long will it take for the value of the account to double? We want to solve for t in the equation

$P=15000e^{(0.04t)}$

$$30{,}000 = 15000e^{0.04t}$$ (divide by 15000)

$$2 = e^{0.04t}$$ (Take ln of both sides)

$$\ln 2 = 0.04\, t$$ $\ln(e^{x})=x$

(solve for t)

$25*\ln 2$ $$\frac{\ln 2}{0.04} = t$$

$$t \cong 17.33$$

So the value of the investment doubles in 17.33 years (about) or 17 years and 4 months. In about 52 years the portfolio will be 8 times its original value.

Remark: Notice the time to double is (ln 2)/0.04 and so does not depend on how much is invested. Time to double in value is T = ln 2/i (i is the intereste rate)

Example: A bond portfolio earns approximately continuously compounded interest at 5% per year. If the portfolio is worth \$60,000 now then how much will it be worth in 2 years? in t years? When will it be worth \$100,000?

$P = 60000e^{it} = 60000e^{0.05t}$ - value of the portfolio after t years.

$P(2) = 600000e^{0.05*2} = 60000\, e^{0.1} = 66{,}310.25$

Solve for t in the equation $100{,}000 = P(t) = 60000e^{0.05t}$

$5/3 = e^{0.05t}$

Take the natural log of both sides
$\ln(5/3) = 0.05*t$

$$\frac{\ln(5/3)}{0.05} = t \quad \text{or } t = 20*\ln 5/3 \cong 10.22$$

After 10.22 the portfolio will be worth 100,000.

Test is on material to here.

Sections 1.7 to 3.3 are on the test.

3.4 Radioactive Decay.

Example: Carbon-14 is a radioactive isotope of carbon. (Most carbon has atomic number 12.) Carbon-14 therefore decays at a rate proportional to the amount present and it has a half life of 5730 years. That is any isolated sample with 1 gram of Carbon-14 will have half a gram of Carbon-14 5730 years later. In particular if a living thing dies today then the amount of C-14 is

$$P(t) = P_0 e^{kt}$$

because C-14 is no longer renewed after death.

Example: A mastadon tusk has 18% of the C-14 that it would have had at death. How long ago did it die?

First of all we must find k: $y = y_0 e^{kt}$ $(y'=ky)$

$\frac{1}{2} y_0 = y_0 e^{5730k}$ (Take ln of both sides.)

or $-\ln 2 = 5730\,k$ $\ln(1/2) = -\ln 2$

$$k = -\frac{\ln 2}{5730}$$

We are now interested in when

$0.18 y_0 = y_0 e^{kt}$ Take ln of both sides

$$\ln 0.18 = kt = -\frac{\ln 2}{5730}\, t$$

$$\text{So } t = -\frac{\ln 0.18}{\ln 2}\, 5730 = 14175.6$$

The mastadon bone is about 14,176 years old

a^x and $\log_a$

$a = e^{(\ln a)}$

So $a^x = e^{(\ln a\, x)}$

Example $2^x = e^{(\ln 2\ \ x)} = e^{(0.693\, x)}$