~~2.7~~

Chapter 3: Exponential and Logarithm Functions.

Exponential Functions. a^n = a^n

2^4 = 2*2*2*2 (multiply 4 times0

(2^4)*(2^3) = 2^7

2^(1/3) is the unique (positive) number such that (2^(1/3))^3=2

$$2^{(-1/3)} = \frac{1}{2^{(1/3)}}$$

In general $2^{(-n)} = \dfrac{1}{2^n}$ for any n

2^(5/3) = (2^(1/3))^5 = (2^5)^(1/3)

We can take 2 to any fractional power with these definitions. The book argues that 2^x makes sense for any real number x because we can approximate x by its finite decimal expansion (which is rational) and take a limit. This is true but not so obvious.

Properties of Exponential Functions: a^x where a>0

1) (a^x)(a^y) = a^(x+y)

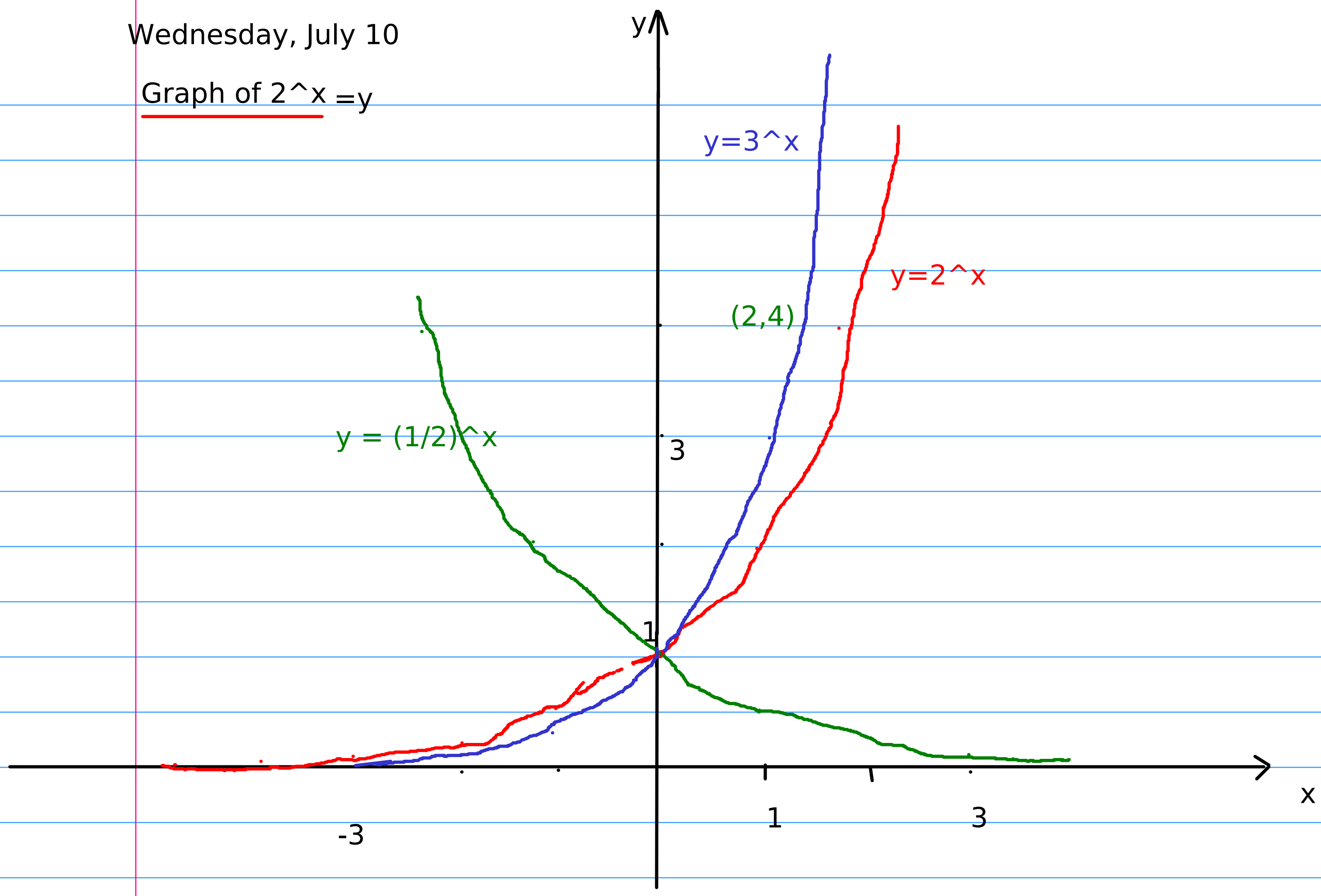
2) a^x/a^y = a^(x-y)

3) (a^x)^y = a^(xy)

4) a^(-x) = 1/a^x

Wednesday, July 10

Graph of 2^x =y



Graph of (1/2)^x = 2^(-x)

Graph of y = 3^x

Note a^x>0 for all x.

Differentiation:

If f(x) = a^x then

$$\frac{f(x+h)-f(x)}{h} = \frac{a^{(x+h)}-a^x}{h} = \frac{a^x * a^h - a^x}{h} = a^x \frac{a^h-1}{h}$$

→ L_a

This says f is differentiable at x if $\lim_{h \to 0} \frac{a^h-1}{h}$ exists. (f is differentiable everywhere if it is differentiable at 0. The limit

$$\lim_{h \to 0} \frac{a^h - 1}{h} = L_a$$

does indeed exist and so f(x) is differentiable and

$$\frac{d}{dx} a^x = L_a \, a^x$$

That is the derivative of a^x is a constant L_a times a^x.
What is L_a? AS a>0 increases L_a increases and it goes from very negative when 0<a<<1 to L_1=0 when a=1 to very positive when a>>1.

Definition: We define e> 1 to be the choice of a for which

$$\lim_{h \to 0} \frac{e^h-1}{h} = 1$$

$L_1 = 0$
$L_2 = 0.69$
$L_3 = 1.03$

2<e <3 $L_e = 1$

Then

$$\frac{d}{dx} e^x = e^x$$

We discover that e = 2.718281828495

e^x is the ``identity of differentiation'' It is the natural exponential

We shall define the ``natural logarithm'' ln x to be the inverse of the function e^x that is

$$\ln x = \log_e x$$

(log base e). So

$$e^{(\ln x)} = x \text{ and } \ln(e^x) = x$$

Then $L_a = \ln(a)$.

$$\frac{d}{dx} a^x = (\ln a) a^x$$

$$\frac{d}{dx} 2^x = (\ln 2) 2^x \approx (0.6931) 2^x$$

Of course ln e = 1 and so

$$\frac{d}{dx} e^x = (\ln e) e^x = e^x$$

Example: Differentiate $y = e^{(3x)}$: $y' = e^{(3x)} 3 = 3 e^{(3x)}$

Recall the chain rule $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Example: Differentiate $y = e^{(x^2)}$ $y' = e^{(x^2)}2x = 2x\, e^{(x^2)}$

Example: Differentiate y = x^2e^x

Recall the product rule. $\frac{d}{dx}(f(x)g(x)) = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$

Therefore y'= e^x(2x)+ x^2e^x= e^x[2x + x^2]= $e^x [2x + x^2]$

Example: Differentiate $y = \frac{e^{2x}}{x^2+3}$

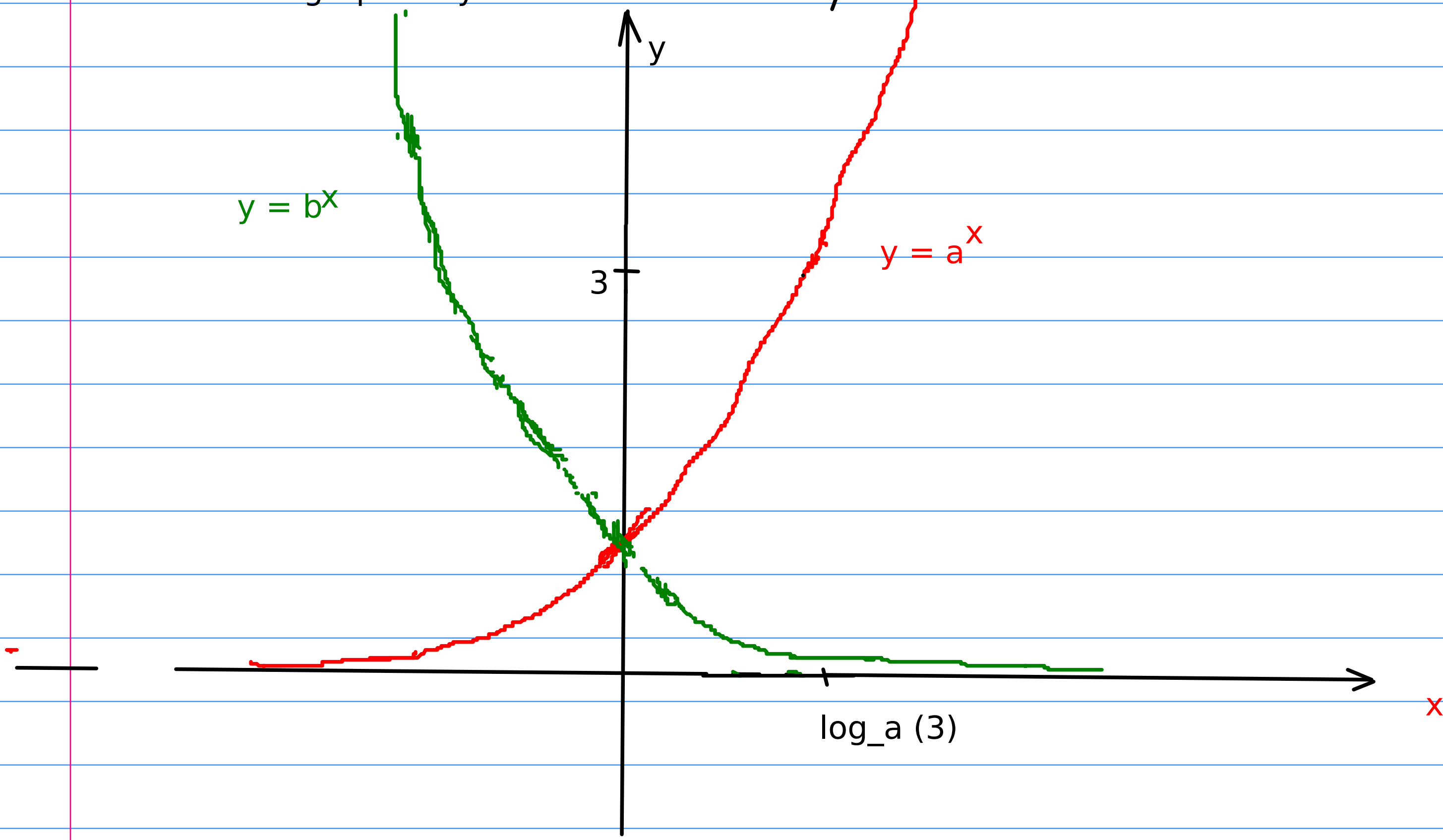
Apply the Quotient Rule $\frac{d}{dx}\frac{N}{D} = \frac{D N' - ND'}{D^2}$

to get N = e^2x What is N'? N'=e^2x *2 = 2e^2x

$$y' = \frac{(x^2+3)2\,e^{(2x)} - e^{(2x)}\,2x}{[x^2+3]^2} = \frac{e^{(2x)}[\,2x^2 - 2x + 6]}{[x^2+3]^2}$$

3.2 Logarithmic Functions

Recall the graph of $y = a^x$ for $a>0$ and $a \neq 1$



0<b<1

1<a

In either case the graph is ``one-to-one'' that is given x there is one and only one y>0 so y = a^x (Similarly if 0<a<1). Consequently we can define a function which takes y to the corresponding x. This is log_a

$$\log_a(a^x) = x \quad \text{and} \quad a^{\log_a x} = x$$

Example $\log_2(4) = 2$

$\log_{10}(1000) = 3$

$\log_3(1/27) = -3$

$\log_2(4) = \log_2(2^2) = 2$; $\log_{10}(1000) = \log_{10}(10^3) = 3$; ;$\log_3(1/27)$

$= \log_3(27^{-1})$ $\log_3(3^{-3}) = -3$

Notation: The Natural Logarithm is $\ln x = \log_e x$

Example $\ln\left(\frac{1}{e^3}\right) = -3$ $\ln(1/e^3) = \ln(e^{-3}) = -3$

Properties of Logarithms

We write log for $\log = \log_a$

a=2 ; log(2^3*2^4) =log(2^7)= 7

log(2^3) = 3 log(2^4) = 4

1) log(xy) = log(x) + log(y)

2) log (x/y) = log(x) - log(y)

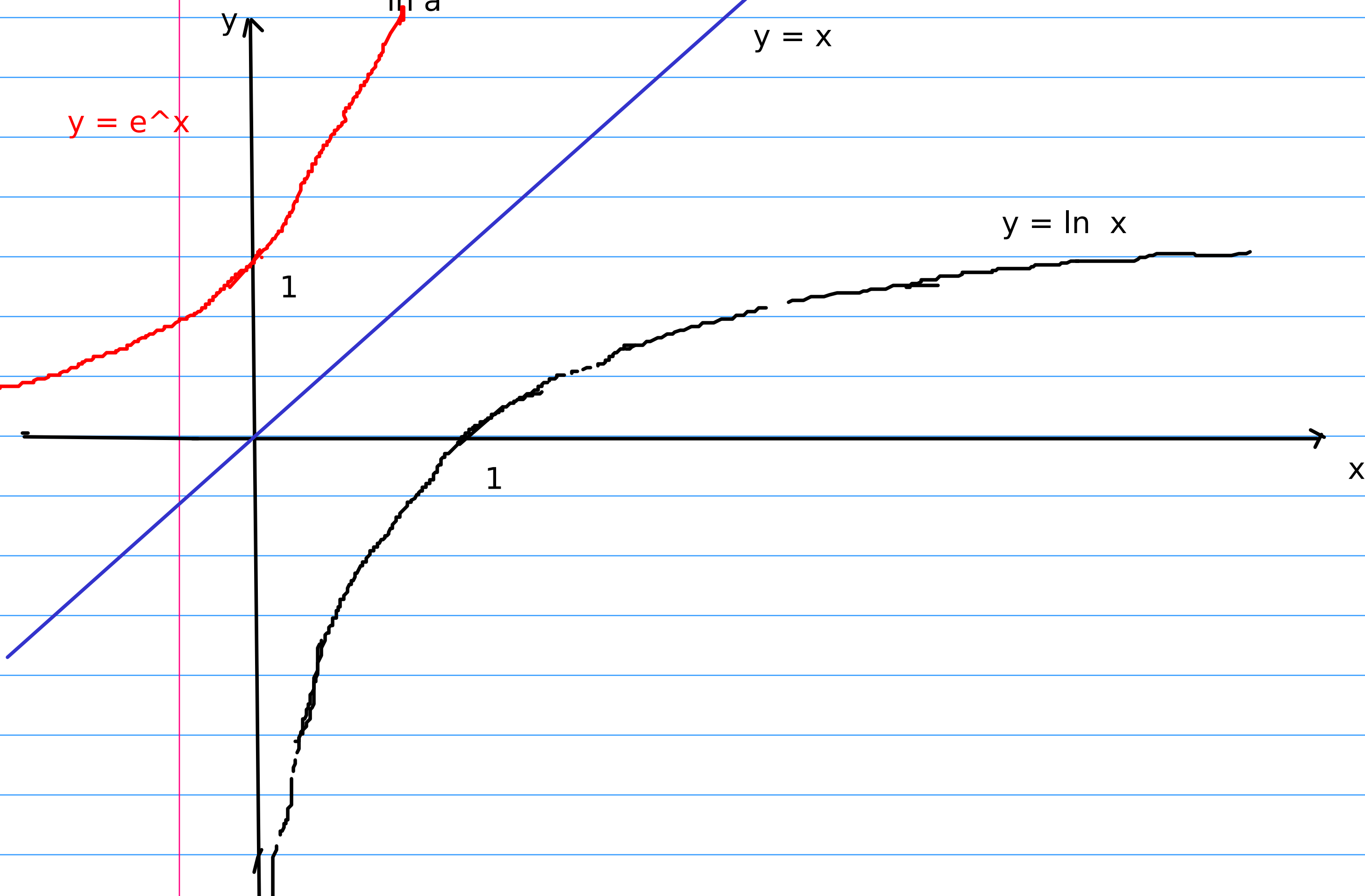
3) $\log (x^k) = k \log(x)$

4) $\log_a a = 1$

5) $\log a^k = k$

6) log 1 = 0

7) $\log x = \frac{\ln x}{\ln a}$ (all logs differ by a multiplicative constant.)



8) $\frac{d}{dx} \ln x = \frac{1}{x}$

Rationale: $e^{\ln x} = x$

Differentiate $e^{\ln x} \frac{d}{dx} \ln x = \frac{d}{dx} x = 1$

chain rule

So $x \frac{d}{dx} \ln x = 1$ or

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

Example 1 Differentiate y = ln (x^2+x +2)

Recall the chain rule: $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Here f(x) = ln x and g(x) = x^2+x+ 2

$y' = \frac{1}{x^2+x+2} (2x+1) = \frac{2x+1}{x^2+x+1}$

Example 2 Differentiate $y = \ln\sqrt{x+1}$

Simplify y = ln (x+1)^(1/2) = $\frac{1}{2} \ln(x+1)$

So $y' = \frac{1}{2} \frac{1}{x+1}\, 1 = \frac{1}{2(x+1)}$

Example Differentiate $y = \ln[x^2(x+1)(x-3)]$

Simplify $y = \ln x^2 + \ln(x+1) + \ln(x-3) = 2\ln x + \ln(x+1) + \ln(x-3)$

so that $y' = \dfrac{2}{x} + \dfrac{1}{x+1} + \dfrac{1}{x-3}$

Example. Differentiate $y = [\ln x]^5$

Does y simplify? $y' = 5[\ln x]^4 (1/x) = \dfrac{5[\ln x]^4}{x}$

3.3 Applications: Uninhibited and Limited Growth Models

We consider quantites y in our world that grow at a rate proportial to themselves

$$y' = ky$$

where k is the constant of proportionality and might be a positive or negative real number.

$y = Ce^{(kx)}$ is an example of such y for any C. All y are of this form.

(x is time)

Examples: 1) Population. The number y of bacteria in a culture grows at a rate proportional to y. It is assumed that the food availability is constant and the death rate is proportional to the population.

2) Radioactive decay. Strontium 90 is a radioactive element and, as such, each atom has a certain probability of decaying into some-thing other than strontium 90 in a given time period. If y is the number of Strontium 90 atomd then $y'=ky$ where $k<0$. The number of strontium 90 atoms is decreasing.

3) A portfolio of investments in bonds grows in value at a

rate proportional to the values of the account or this is a close approximation

If the interest rate is 3% then the value of the is y where

$$y'= (1.03)y$$

roughly. (This approximates compound interest when the compounding period is very short: ``continuous compounding.)

Example: A bond portfolio earns approximately continuously compounded interest at 4% per year. If the portfolio is worth $60,000 now then how much will it be worth in 2 years? in t years?