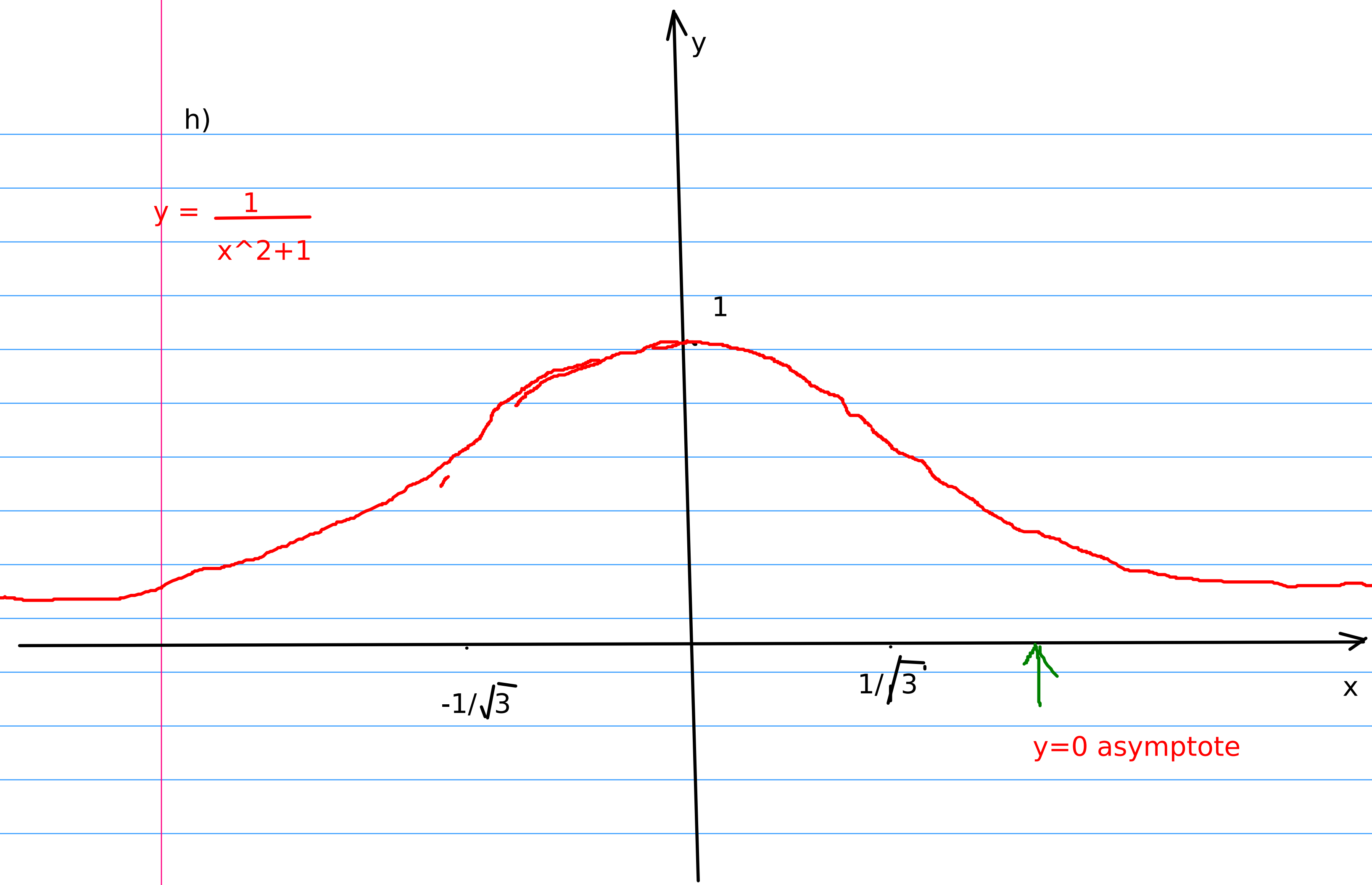
h)

$y = \frac{1}{x^2+1}$



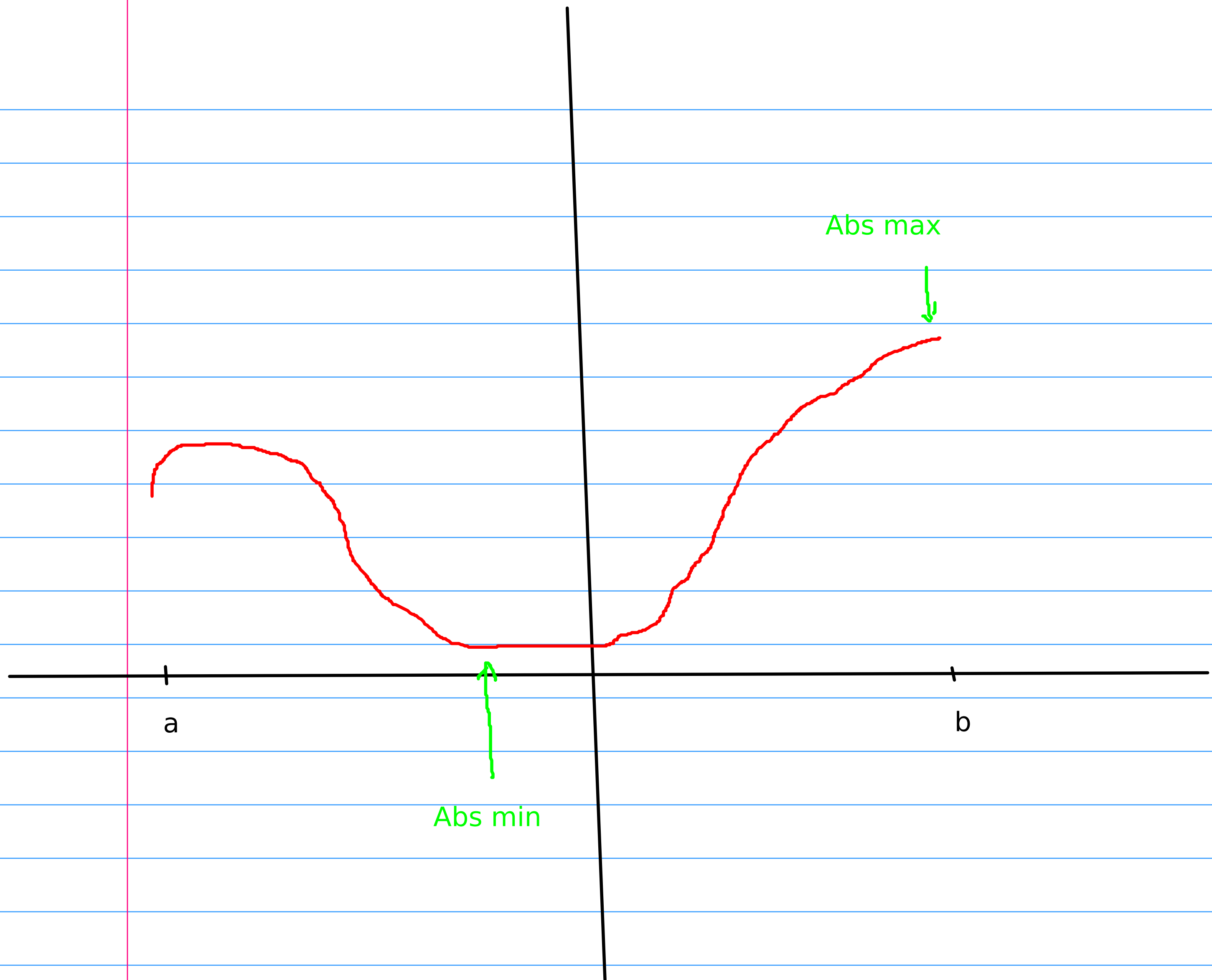
2.4 The case of a continuous function f(x) defined on a closed bounded interval [a,b]

WE look for the ABSOLUTE max and min of f on [a,b].

M is an absolute maximum of f(x) on [a,b] if
minimum

$f(x) \leq M$ for all x, $a \leq x \leq b$ and there is c so that $f(c) = M$

$\geq$



Section 2.4

Theorem Every continuous function defined on a closed and bounded interval [a,b] has both an absolute maximum and minimum.

Max-Min Principle 1 (Theorem 8)

The absolute max and the absolute min of a continuous function defined on a closed bounded interval [a,b] occur either at a critical point or at an end point (a or b).

Example Find the absolute max and min of f(x) = x^2 - 2x, $0 \leq x \leq 3$

1) Check for critical points f'(x) =2x-2 =2(x-1)
Set f'(x) = 0 x=1

f'(x) DNE for some x? No

2) What are the end points? x=0, x=3

3) Evaluate f(x) at each of the points in parts 1 and 2

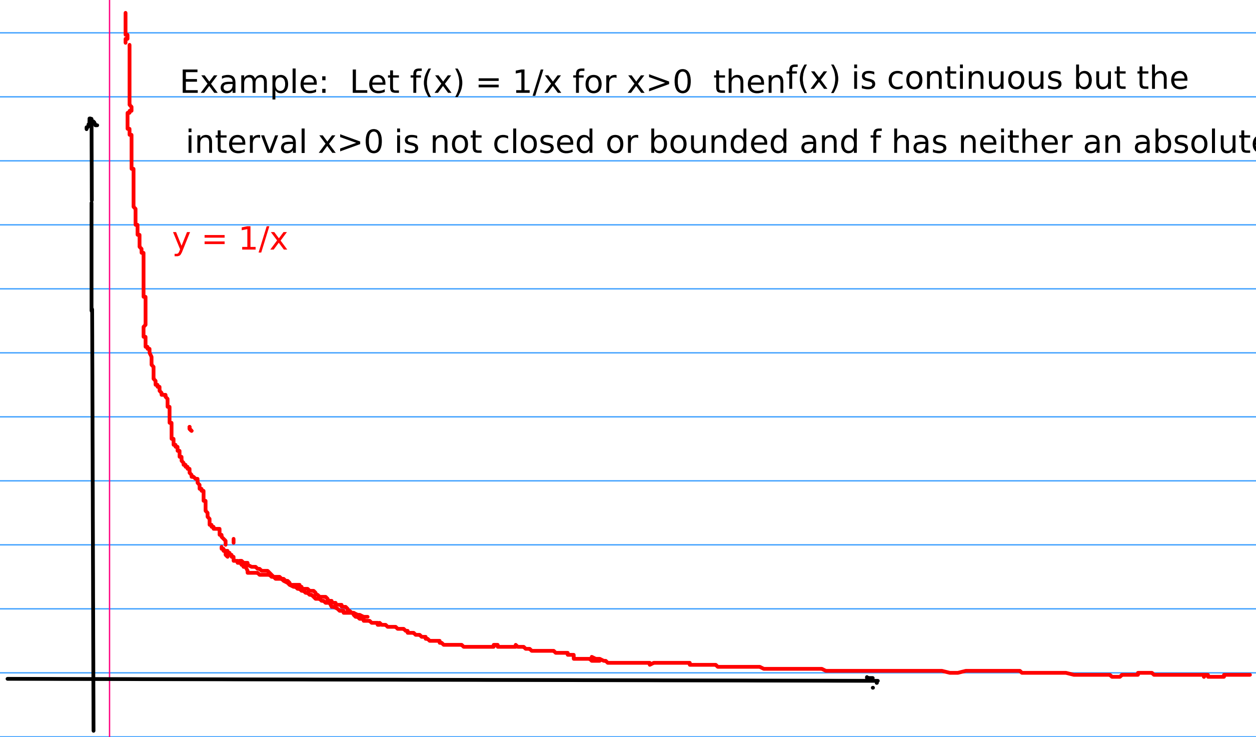
f(x) = x^2-2x

f(1) =1-2 =-1 ← Absolute min value of -1 at x=1

f(0) =0

f(3) = 9-6=3 ← Absolute max value of 3 at x= 3

Example: Let f(x) = 1/x for x>0 then f(x) is continuous but the interval x>0 is not closed or bounded and f has neither an absolute max nor min.



Monday, July 8

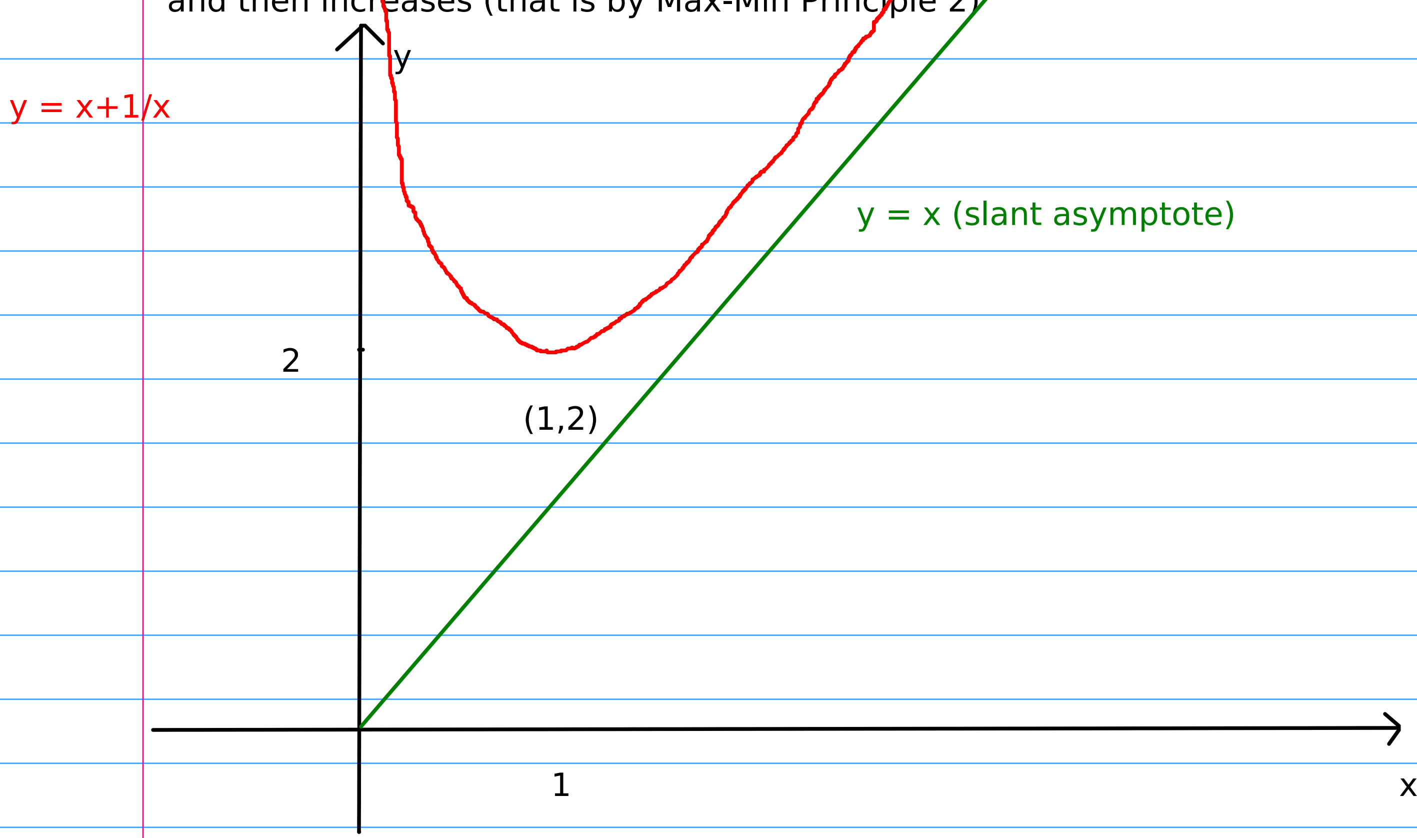
Max-Min Principle 2 (Theorem 9)

If f is differentiable on an entire interval I and if there is exactly one c in I so that $f'(c)=0$ and if c is a relative max (resp. min) then it is an absolute max (resp. min) on I.

Example: Let $f(x) = x + 1/x$, $x>0$. Find the absolute max and min if they exist.

Since the interval $x>0$ is not closed (or bounded) there is no guarantee of an absolute max or min existing.

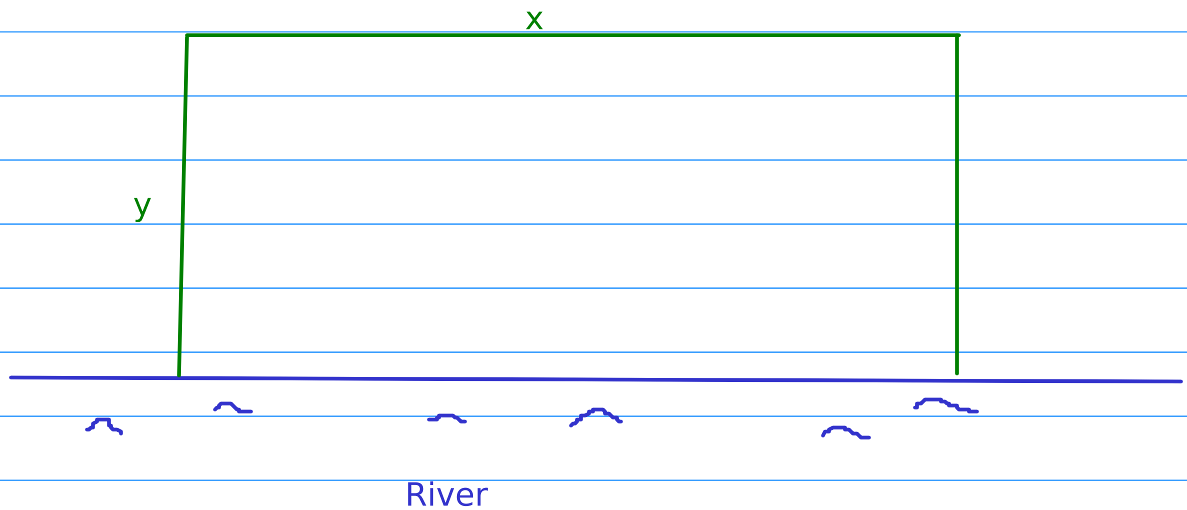
Differentiate $f(x) = x + x^{-1}$ to get $f'(x) = 1 - x^{-2}$. Check for critical points $f'(x) = 0$: $1 - x^{-2} = 0$ so $x^2=1$ so $x = \pm 1$ but $x>0$ so $x=1$.
f is differentiable on the entire interval $x>0$. Is $x=1$ a relative max or min or neither? $f''(x) = 2x^{-3} > 0$ and so $x=1$ is a relative min and so it (second deriv test) is an absolute min. However f(x) has no absolute max because it decreases and then increases (that is by Max-Min Principle 2)



2.5 Max-Min Problems; Business and Economics Applications

Example: A farmer wants to fence off a rectangular plot of land along a river for pasture. She has 600 meters of fence and no fence is needed along the river. What is the largest area that she can enclose?

Solution: Picture



We want to maximize area which for a rectangle is A = xy = x*y

We know that there is 600 meters of fence: x+2y = 600
x = 600-2y

Eliminate all but one variable. A = (600-2y)y = 600y - 2y^2

Find the absolute max / min of A on the closed interval $0 \le y \le 300$
using calculus [0,300] is a closed and bounded interval

A' = 600 - 4y

Check for critical points A'=0: 600-4y = 0 or 600 = 4y or 150 = y
A' is defined for all y and so this is the only critical point.

End Points y = 0 and y = 300
y=0 A= 0

y= 300, A = 600*300 - 2*300*300 =0

Absolute max

A when y = 150 is 600*150 - 2*150*150 = 150*(600 -2*150) =150*300=45000
A when y = 0 or y = 300 is A=0 ←

A ABsolute Min

The farmer should fence off a rectangular portion which is 150 by 300 m of pastureland along the river.

Example (page 274, Number 32 of the text)

Gritz-Charlston is a 300-unit luxury hotel. All rooms are occupied when the hotel charges \$80 per day for a room. For every increase of x dollars in the daily room rate , there are x rooms vacant.Each occupied room costs \$22 per day to service and maintain. What should the hotel charge to maximize profit?

Let p(x) = the price of a room per night when there are x rooms vacant.

p(0) = 80
p(1) = 81
p(x) = 80 + x

300-x is the number of occupied rooms
Revenue is (80+x)(300-x)= 24000+220x-x^2
Costs are 22(300-x) = 6600-22x so that Profit (Revenue minus cost) is

P(x) = 24000 +220x -x^2 -(6600-22x)= 17400 +242x-x^2

242-2x=0
242=2x
121=x

We want to find the price that maximizes the profit on the interval $0 \le x \le 300$.
Find P'= 242-2x so that there is a critical point at x =121 and nowhere else because P'(x) is defined everywhere.
Evaluate P at the endpoints x=0 and x = 300 and at the critical point x=121

P(0) = 17,400
P(300) = 0 Absolute Min
P(121) = 17400 +242(121) -(121)^2= 17400+14641=32041Absol Max

So if the charge is 80+x = 201 dollars per night then the profit is maximized at \$32,041 and there will be 121 vacant rooms.

Max-min principle 2 can be used as well

2.6 Marginals and Differentials.

1) Marginals. If x units of a product are produced then R(x) denotes the revenue C(x) the cost and P(x) the profit.

The Marginal Revenue is R'(x)
The Marginal Cost is C'(x)
The Marginal profit is P'(x)

For example if P'(x) (the marginal profit) is positive then it is worthwhile to think of increasing production. (P(x+1)-P(x) = P'(x)) $P'(x) = \lim_{h\to 0} \frac{P(x+h)-P(x)}{h}$

Example: the Cost of installing x residential swimming pools is C(x)= 22,000x +5x^2 because equipment and costs for installing a pool next door is $22,000 but when more and more pools are installed then there is an additional transportation cost.

Here C'(x) =22,000 +10x so that once 1,000 pools are installed (for example) the next pool will cost approximately $32,000 to install.

marginal cost

2) Differentials.

If f(x) is a differentiable function then the differential of f is defined to be

$$df = f'(x)\,dx$$

Intuitively the differential records how fast f(x) is changing relative to x.

Example: Suppose that a shipping container is a perfect cube but it is made by hand and the side length may vary: the side length is 6 +/- 0.1 feet. Use differentials to the variation in volume.

V(x) = x^3 x is the sidelength of the container.

dV = V'(x) dx or dV = 3x^2 dx.

V(6) = 6^3 = 216.

The variation ΔV is approximately dV = 3x^2 dx ~= 3(6)^2 Δ dx =108(+/- 0.1)
=+/- 10.8

The error is using 216 cu ft as the approximation of volume is about 10.8 cu ft.

((6.1)^3 - 6^3 =10.981) V = 216 +/- 10.8

(5.9)^3-216 ≠ -10.981

Mean Value Theorem: If f(x) is continuous on [a,b] and differentiable on (a,b) then there exists c, a<c<b so that

f(b)-f(a) = f'(c)(b-a)

(The change in f is the derivative times the change in x.) If the interval (a,b) is very small we can approximate f'(c) by f'(a)
f(b)-f(a) = f'(a)(b-a)

Or $\Delta f \cong f'(a)\Delta x$ Δ Delta

The approximation improves as Δx gets small and we write df = f'(x)dx

Example: Find dy if

1) $y = \sqrt{5x+3}$

2) $y = \dfrac{4}{3t+9}$

Solution 1) Here y = (5x+3)^(1/2) do that $y' = \frac{1}{2}(5x+3)^{(-1/2)}\, 5$ by the generalized power rule. Therefore

$$dy = \frac{5}{2}(5x+3)^{(-1/2)}dx$$

$$2)\ y = \frac{0 - 4(3)}{(3t+9)^2} = -\frac{12}{(3t+9)^2}$$

Quotient Rule

$$\frac{DN' - ND'}{D^2}$$

Therefore

$$dy = -\frac{12}{(3t+9)^2}\,dt$$

Example if y = (3x+1)^(-1/3) then y' = (-1/3)(3x+1)^(-4/3) *3= -(3x+1)^(-4/3)

Differential notation: dy = -(3x+1)^(-4/3) dx

If, for example x = 21 the 3x+1 =64 so -(3x+1)^(-4/3) = -64^(-4/3) = $-\frac{1}{256}$

if x =21 then df = $-\frac{1}{256}$ dx

~~2.7~~

Chapter 3: Exponential and Logarithm Functions.

Exponential Functions. a^n = a^n a>0

a=2

2^4 = 2*2*2*2 (multiply 4 times)

(2^4)*(2^3) = 2^7

2^(1/3) is the unique (positive) number such that (2^(1/3))^3=2

$$2^{(-1/3)} = \frac{1}{2^{(1/3)}}$$

In general $2^{(-n)} = \frac{1}{2^n}$ for any n

2^(5/3) = (2^(1/3))^5 = (2^5)^(1/3)

We can take 2 to any fractional power with these definitions. The book argues that 2^x makes sense for any real number x because we can approximate x by its finite decimal expansion (which is rational) and take a limit. This is true but not so obvious.

Properties of Exponential Functions: a^x where a>0

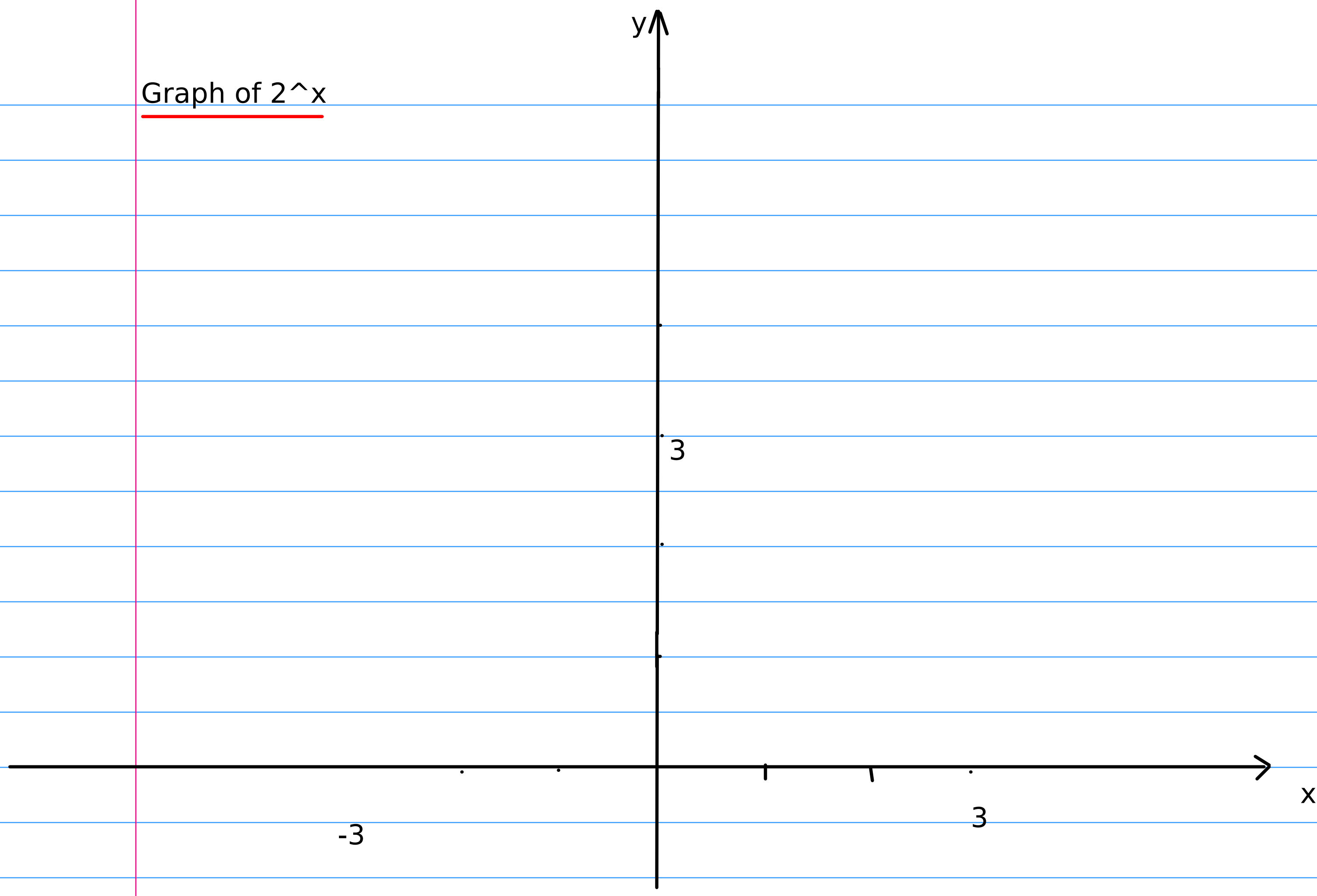
1) (a^x)(a^y) = a^(x+y)

2) a^x/a^y = a^(x-y)

3) (a^x)^y = a^(xy)

4) a^(-x) = 1/a^x

Graph of 2^x



Graph of $(1/2)^x = 2^{(-x)}$

Graph of $y = 3^x$

Note $a^x > 0$ for all x.

Differentiation:

If $f(x) = a^x$ then

$$\frac{f(x+h)-f(x)}{h} = \frac{a^{(x+h)}-a^x}{h} = \frac{a^x * a^h - a^x}{h} = a^x \frac{a^h-1}{h}$$

This says f is differentiable at x if $\lim_{h \to 0} \frac{a^h-1}{h}$ exists. (f is differentiable everywhere if it is differentiable at 0. The limit

$$\lim_{h \to 0} \frac{a^h - 1}{h} = L_a$$

does indeed exist and so f(x) is differentiable and

$$\frac{d}{dx} a^x = L_a a^x$$

That is the derivative of a^x is a constant L_a times a^x.
What is L_a? AS $a>0$ increases L_a increases and it goes from very negative when $0<a<<1$ to $L_1=0$ when $a=1$ to very positive when $a>>1$.

Definition: We define $e> 1$ to be the choice of a for which

$$\lim_{h \to 0} \frac{e^h-1}{h} = 1$$

Then

$$\frac{d}{dx} e^x = e^x$$

We discover that e = 2.718281828495

e^x is the ``identity of differentiation'' It is the natural exponential

We shall define the ``natural logarithm'' ln x to be the inverse of the function e^x that is

$$\ln x = \log_e x$$

(log base e). So

$$e^{(\ln x)} = x \text{ and } \ln(e^x) = x$$

Then $L_a = \ln(a)$.

$$\frac{d}{dx} a^x = (\ln a)\, a^x$$

$$\frac{d}{dx} 2^x = (\ln 2)\, 2^x \cong (0.6931)\, 2^x$$

Example: Differentiate $y = e^{(3x)}$: $y' = e^{(3x)}\, 3$

Recall the chain rule $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Example: Differentiate $y = e^{(x^2)}$ $y' =$

Example: Differentiate $y = x^2 e^x$

Recall the product rule. $\frac{d}{dx}(f(x)g(x)) = g(x)\frac{d}{dx} f(x) + f(x)\frac{d}{dx} g(x)$