2.3 Graph Sketching: Asymptotes and Rational Functions.

A rational function is the ratio of two polynomials

Examples 1) $f(x) = \dfrac{2x+4}{x+1}$

2) $f(x) = \dfrac{x+1}{x^2+3}$

3) $g(x) = \dfrac{4x^2+5}{x^2+5x+6}$

4) $h(t) = \dfrac{5t^3+t^2+2t}{t^2-3t+2}$

Vertical Asymptotes correspond to points x where there is a division by 0

Example: 1) A vertical asympote for the curve y = f(x) is x=-1

2) No vertical asymptote because x^2+3 is never 0

3) $x^2+5x+6 = (x+2)(x+3)$ x=-2 and x=-3 are the two vertical asymptotes

4) $t^2-3t+2 = (t-2)(t-1)$ t=1 and t=2 are vertical asymptotes

Section 2.3 Horizontal Asymptote to y=f(x) correspond to a straight line y = constant so that the curve gets close to the line for large x (positive or negative)

For rational functions factor out the highest power downstairs

$$\lim_{x\to\infty} \frac{1}{x} = 0 \qquad \lim_{x\to-\infty} \frac{1}{x} = 0$$

Example 1) $f(x) = \dfrac{2x+4}{x+1} = \dfrac{\cancel{x}}{\cancel{x}} \dfrac{(2+4/x)}{(1+1/x)}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2+4/x}{1+1/x} = \frac{2+0}{1+0} = 2$$

Horizontal Asymptote is y = 2 at $x = \infty$ and at $x = -\infty$

So y=2 is a horizontal asymptote.

Taking the limit as x goes to $-\infty$ gives the same value y=2 is a horizontal asymptote at +/- ∞

Example 2) 2) $f(x) = \dfrac{x+1}{x^2+3}$ (degree on bottom larger)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+1}{x^2+3} = \lim_{x\to\infty} \frac{\cancel{x^2}\left(1/x + 1/x^2\right)}{\cancel{x^2}\left(1 + 3/x^2\right)} = \lim_{x\to\infty} \frac{\overset{\to 0}{1/x} + \overset{\to 0}{1/x^2}}{1 + \underset{\to 0}{3/x^2}} = \frac{0}{1} = 0$$

So the horizontal asymptote is y = 0

(Same as x approaches - ∞)

Example 3) $g(x) = \dfrac{4x^2+5}{x^2+5x+6}$ (degree on bottom equals degree on top)

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{\cancel{x^2}}{\cancel{x^2}} \frac{4+5/x^2}{1+5/x+6/x^2} = \lim_{x\to\infty} \frac{4+5/x^2}{1+5/x+6/x^2} = \frac{4+0}{1+0+0} = 4$$

So y = 4 is a horizontal asymptote

Example 4) $h(t) = \dfrac{5t^3 + t^2 + 2t}{t^2-3t+2}$ (degree on bottom is one smaller than one on top

$$h(t) = \frac{\cancel{t^2}}{\cancel{t^2}} \frac{5t+1+2/t}{1-3/t+2/t^2} = \frac{5t+1+\overset{\to 0}{2/t}}{1-\underset{\to 0}{3/t}+\underset{\to 0}{2/t^2}} \approx \frac{5t+1+0}{1-0}$$

So y = 5t+1 is an oblique asymptote.

If the degree on top is 2 or more larger than that on the bottom then this doesn't work. There is no horizontal or oblique asymptote

Example $y = \dfrac{x^3}{x+1}$ does not have a horizontal or oblique asymptote

Wednesday, July 3

Example: Graph $y = \dfrac{x}{x-2}$

Strategy a) intercepts Set x=0 to find the y -intercept (here y = 0); Set y=0 to get the x-intercept

b) Asymptotes (and domain)

c) Derivative and domain

d) Critical values (f'(c)=0 or DNE)

e) Intervals of Increase and Decrease. Relative Max/Min

f) Inflection points and points where f"(x) DNE

g) intervals of concavity

h) Sketch (Table of Values?)

Solution

a) (0,0)

b) Vertical Asymptote x=2 and so x =2 is not in the domain

Horizontal asymptote: $\lim\limits_{x \to \infty} \dfrac{\cancel{x}}{\cancel{x}} \dfrac{1}{1-2/x} = \lim\limits_{x \to \infty} \dfrac{1}{1-2/x} = \dfrac{1}{1-0} = 1$

So y=1 is a horizontal asympote.

$y = \dfrac{x}{x-2}$

c) $y' = \dfrac{(\cancel{x}-2)-\cancel{x}}{(x-2)^2} = \dfrac{-2}{(x-2)^2} = -2(x-2)^{(-2)}$

$\dfrac{N'D - D'N}{D^2}$

$y'' = -2(-2)(x-2)^{(-3)} = \dfrac{4}{(x-2)^3}$

d) Critical points. Set y'=0 : $\dfrac{-2}{(x-2)^2} = 0$ No such x

y' does not exist at x=2 but y does not either and so x=2 is not called a critical point. No critical points

$y' = \frac{-2}{(x-2)^2}$

e) Intervals of increase and decrease: $(-\infty, 2)$ $y'(0) = -1/2 < 0$ Decreasing

$(2, \infty)$ $y'(3) = -2 < 0$ Decreasing

f) $y'' = \frac{4}{(x-2)^3}$ Set y" = 0 but there is no solution. No inflection points.

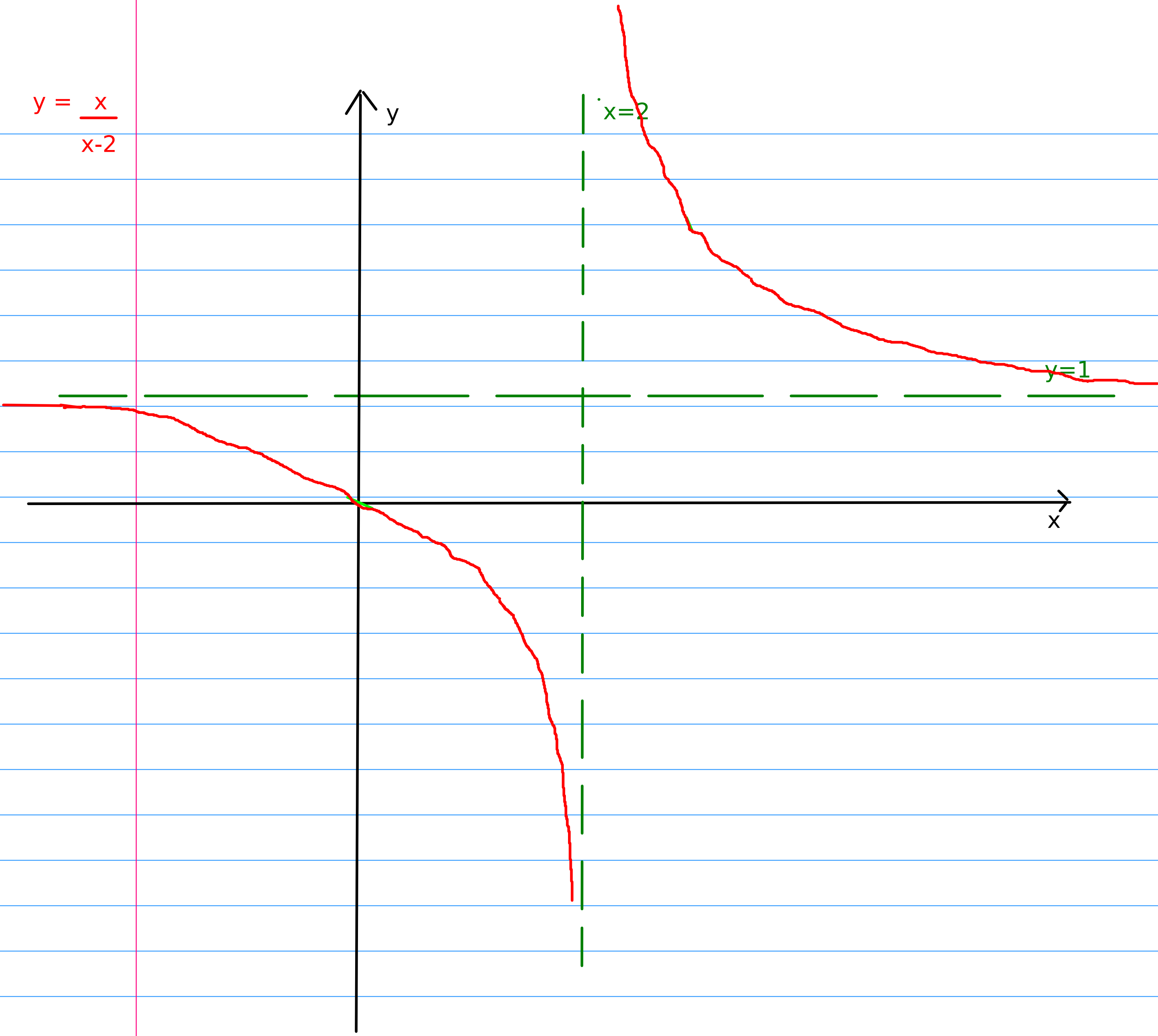
g)
Intervals of Concavity: $(-\infty, 2)$ $y''(0) = -1/2 < 0$ (concave down)

$(2, \infty)$ $y''(3) = 4 > 0$ (concave up)

h) Table of values

x	y	y'	y"
0	0	-1/2	<0
1	-1	-2	<0
2	DNE		
3	3	-2	>0

$y = \frac{x}{x-2}$



Example: Graph $y = \dfrac{1}{x^2+1}$

Solution: a) y-intercept y= 1: (0,1) is on the graph

x-intercept (Set y = 0 to get 0=1) no x-intercept.

b) Vertical Asymptote? Division by 0 ? No vertical asymptote

Horizontal asymptote? y= 0

$$\frac{1}{x^2+1} = \frac{\cancel{x^2}}{\cancel{x^2}} \; \frac{1/x^2}{1+1/x^2}$$

$$\lim_{x\to\infty} \frac{1}{x^2+1} = \lim_{x} \frac{1/x^2}{1+1/x^2} = \frac{0}{1+0} = 0 \qquad y=0$$

c) $y' = \dfrac{-2x}{(x^2+1)^2}$ (extended power rule)

y = (x^2+1)^(-1) y' = (-1)(x^2+1)^(-2) 2x

$$y'' = -2 \; \frac{(x^2+1)^{\cancel{2}} - x(2)\cancel{(x^2+1)}(2x)}{(x^2+1)^{\cancel{4}\,3}}$$ (Quotient Rule)

$$\frac{N'D \; - \; ND'}{D^2}$$

$$= -2 \; \frac{1-3x^2}{(x^2+1)^3}$$

$$= 2 \; \frac{3x^2-1}{(x^2+1)^3}$$

So that y"=0 when x = ____________

d) Critical Points y'=0 or y' DNE

y'=0 implies x = 0 $\dfrac{-2x}{(x^2+1)^2} = 0$ so -2x =0

y' DNE? No

x=0 is the only critical point.

e) Intervals of Increase/Decrease

$$y' = \frac{-2x}{(x^2+1)^2}$$

$(-\infty, 0)$ $y'(-1) = 2/4 > 0$ Increasing

$(0, \infty)$ $y'(1) = -2/4 < 0$ decreasing

At x=0, y =1 Is x=0 a relative max or min or neither? Max

f) y"=0?

$$y'' = 2\frac{3x^2-1}{(x^2+1)^3}$$

y"=0? 3x^2-1=0 x^2 = 1/3 so that x = +/- $1/\sqrt{3}$

y" DNE? No

g) Intervals of Concavity

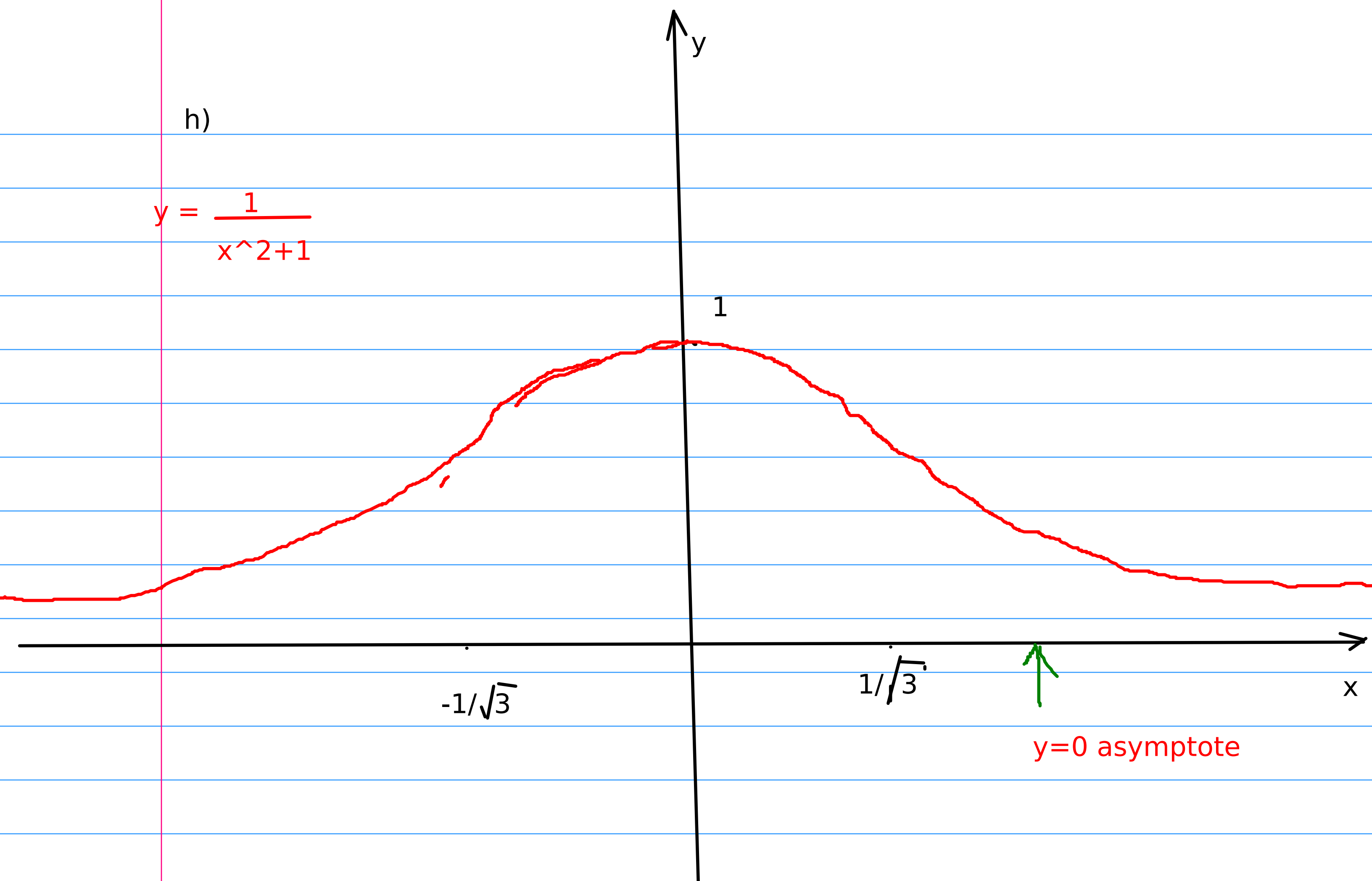
$(-\infty, -1/\sqrt{3})$ $y''(-1) = 2\frac{2}{8} = 1/4 > 0$ ∪

$(-1/\sqrt{3}, 1/\sqrt{3})$ $y''(0) < 0$ ∩

$(1/\sqrt{3}, \infty)$ $y''(1) > 0$ ∪

h)

$y = \frac{1}{x^2+1}$



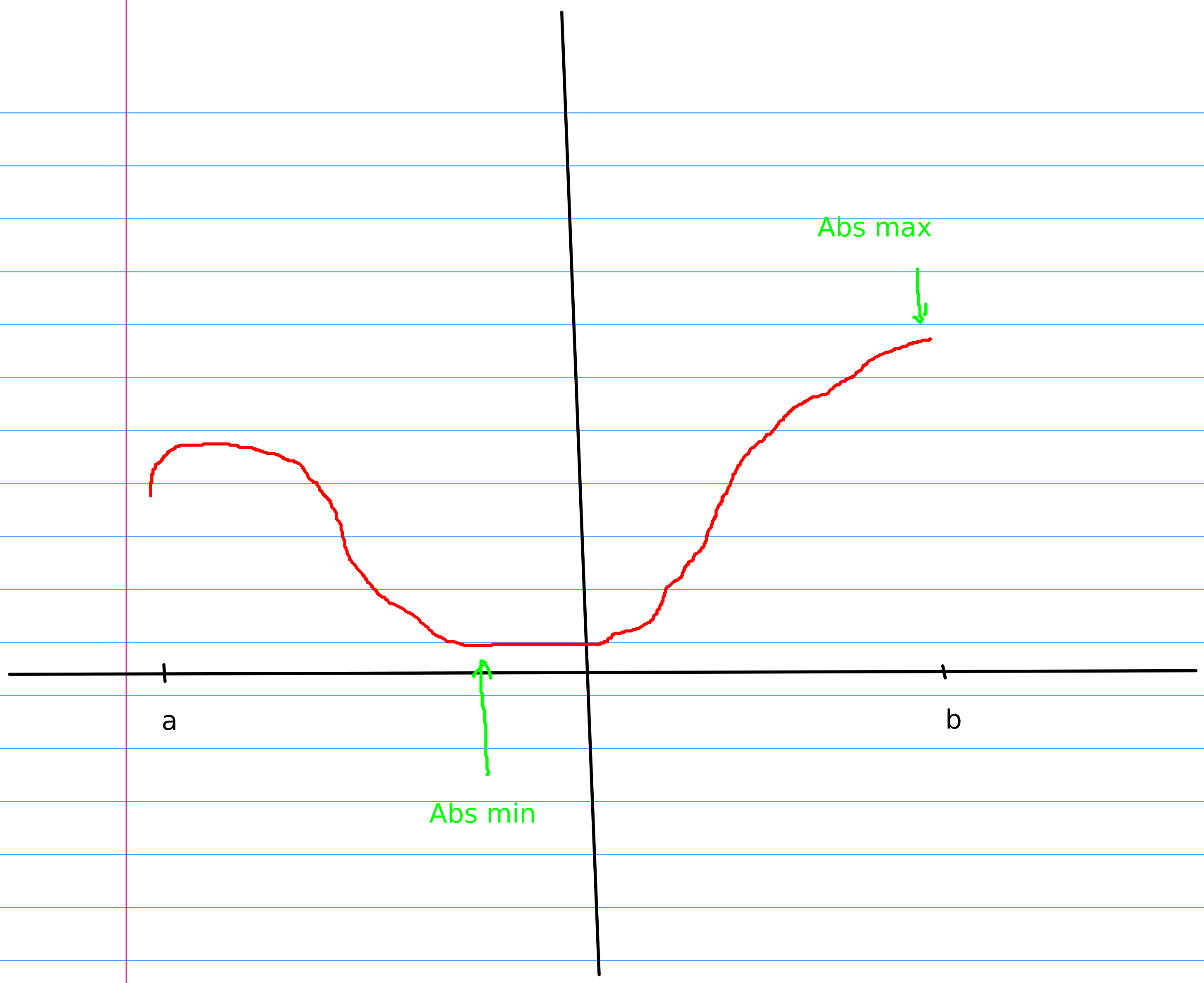
2.4 The case of a continuous function f(x) defined on a closed bounded interval [a,b]

WE look for the ABSOLUTE max and min of f on [a,b].

M is an absolute maximum of f(x) on [a,b] if
minimum

$f(x) \leq M$ for all x, $a \leq x \leq b$ and there is c so that $f(c) = M$

$\geq$



Theorem Every continuous function defined on a closed and bounded interval [a,b] has both an absolute maximum and minimum.

The absolute max and the abolute min occur either at a critical point or an end point (a or b).

Example Find the absolute max and min of f(x) = x^2 - 2x, $0 \leq x \leq 3$

1) Check for critical points f'(x) =2x-2 =2(x-1)
Set f'(x) = 0 x=1

f'(x) DNE for some x? No

2) What are the end points? x=0, x=3

3) Evaluate f(x) at each of teh points in parts 1 and 2

$f(x) = x^2-2x$

$f(1) = 1-2 = -1$ ← Absolute min value of -1 at x=1

$f(0) = 0$

$f(3) = 9-6 = 3$ ← Absolute max value of 3 at x= 3