Section 2.1 in the Text

Theorem 3 First Derivative Test for Relative Extrema:

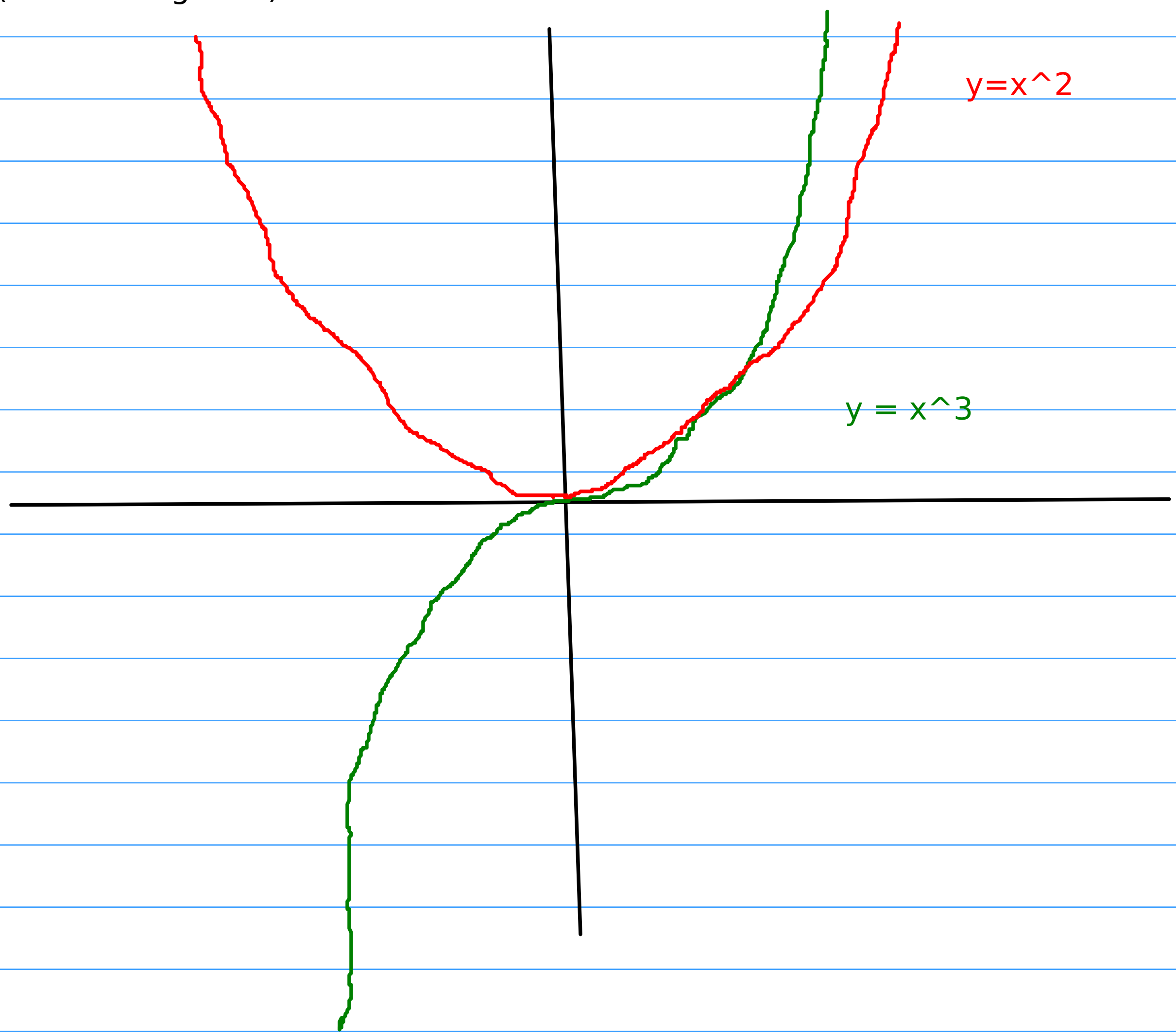
Suppose that c is the only critical point of $f(x)$ in some interval $(a,b)$. $a<c<b$

1) If $f'(x)<0$ for $x<c$ and $f'(x)>0$ for $c<x$ then c is a relative min
2) If $f'(x)>0$ for $x<c$ and $f'(x)<0$ for $c<x$ then c is a relative max
3) Otherwise $f'(x)$ does not change sign at c and c is not a relative extrema.

x=0 is a critical point

Example $f(x) = x^2$ has a critical point when $f'(x) = 2x = 0$, and that is a relative min because $f'(x) = 2x$ goes from negative to positive at $x=0$.

Example $f(x) = x^3$ has a critical point when $f'(x) = 3x^2 = 0$ and that is $x=0$ but that is not a relative max nor min because $f'(x)$ does not change sign (it is nonnegative) at $x = 0$.



Example: Find the relative extrema of $f(x) = 4x^3 - 3x^4$

Solution: $f'(x) = 12x^2 - 12x^3 = 12x^2(1-x)$

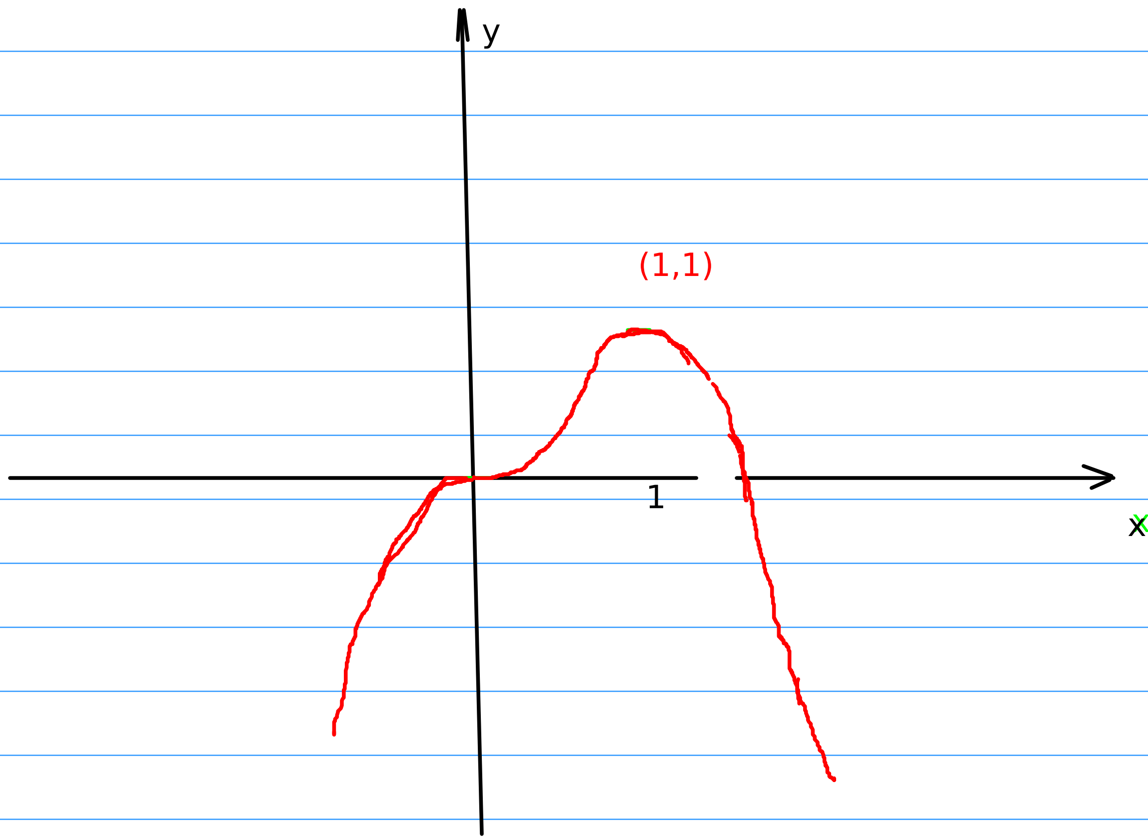
The critical points are where $f'(x) = 0$ and that is at $x = 0$ and $x = 1$.

There are no points where $f'(x)$ does not exist and so there are no other critical points.

Near $x = 0$, $f'(x)>0$ so that $x=0$ is neither a local max nor min.

At $x = 1$ $f'(x)$ goes from being positive to being negative as x increase through 1

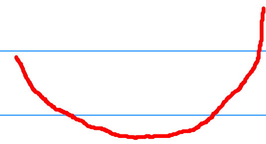
($f'(x)>0$ if $x<1$ and $f'(x)<0$ if $x>1$.) Therefore $x = 1$ is a relative max .



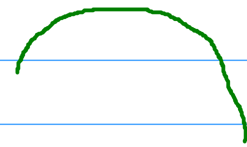
Monday, July 1

## 2.2 Using Second Derivatives.

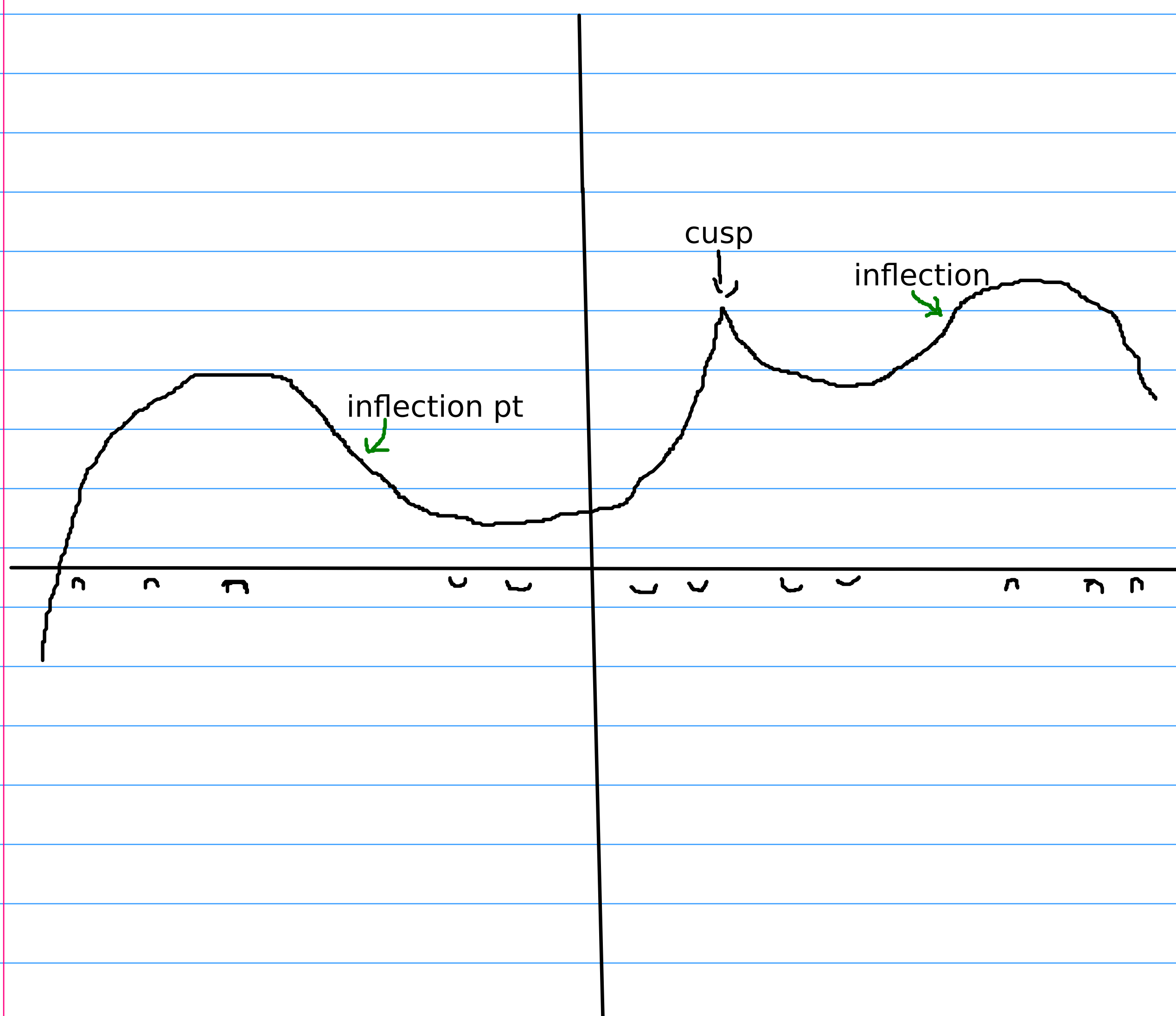
If $f''(x) > 0$, $a < x < b$ then f is concave up



If $f''(x) < 0$, $a < x < b$ then f is concave down.



On which intervals is the function, graphed below, concave up and concave down.?



Definition:  If f"(a) = 0 then (a,f(a)) is said to be a point of inflection.

If there is a transition from concave up to down or vice versa at a then a is an
f"(a)=0 inflection point or f"(a) does not exist.

Example: Graph y = 6x^2- x^4.  Indicate relative max and min.

Check that  y' = 12x -4x^3 = 4x(3 - x^2)
y" = 12 - 12x^2 = 12(1-x^2)=12(1-x)(1+x)

y"(x) = 0 at x = +/-1 If x<-1 then y"<0
The inflection points are (-1,5) and (1,5).  Note y" is defined everywhere.
Find the intervals of concavity

| | | | |
|---|---|---|---|
| $(-\infty,-1)$ | y"(-2)=12(3(-1)<0 | concave down | $\cap$ |
| (-1,1) | y"(0) =12>0 | concave up | $\cup$ |
| $(1,\infty)$ | y"(2) =12(-1)(3)<0 | concave down | $\cap$ |

Now lets graph.  We see from looking at y' that the critical points are

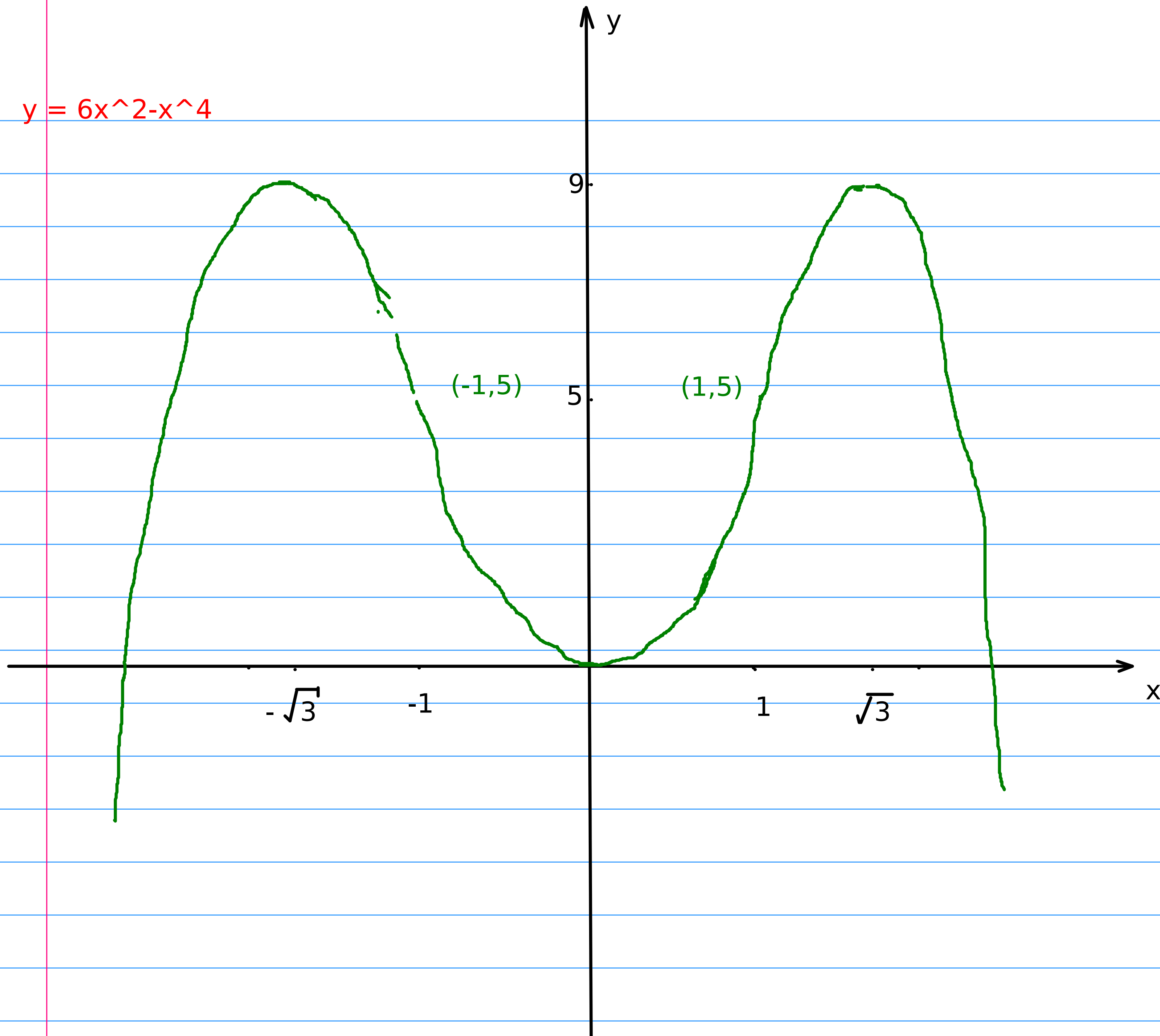
x =  0,+/-$\sqrt{3}$ y=6x^2-x^4 y' = 4x(3-x^2)

| | | | |
|---|---|---|---|
| $(-\infty,-\sqrt{3})$ | y'(-2)=4(-2)(3-4)=8>0 | Increasing | $-\sqrt{3}$ is a rel max |
| $(-\sqrt{3},0)$ | y'(-1) = 4(-1)(3-1)<0 | Decreasing | |
| $(0,\sqrt{3})$ | y'(1) = 41(3-1)>0 | Increasing | |
| $(\sqrt{3},\infty)$ | y'(2) =4(2)(3-4)<0 | Decreasing | |

0 is a relative min and $\sqrt{3}$ is a relative min

Now graph

$y = 6x^2 - x^4$




| x | y | y' | y'' |
|---|---|---|---|
| -2 | 8 | 8 | <0 |
| $-\sqrt{3}$ | 9 | 0 | <0 |
| -1 | 5 | -8 | 0 |
| 0 | 0 | 0 | >0 |
| 1 | 5 | 8 | 0 |
| $\sqrt{3}$ | 9 | 0 | <0 |
| 2 | 8 | -8 | <0 |

Theorem 5: The Second Derivative Test for Relative Extrema

Suppose that f is differentiable at every point of an interval (a,b) and f'(c)=0 for some c, a<c<b.

1. If f"(c)>0 then (c,f(c)) is a relative minimum

2. If f"(c)<0 then (c,f(c)) is a relative maximum

If f"(c)=0 or f"(c) does not exist then the first derivative test might be appropriate.

Example. In the previous example f(x) = 6x^2-x^4 we see that the critical points are (f'(x) =4x(3-x^2)) x=0, x = +/$\sqrt{3}$

f"(x) =12(1-x)(1+x)

At x=0, f"(x)=12>0 so (0,0) is a relative minimum

At x = -$\sqrt{3}$ f"(-$\sqrt{3}$) < 0 so that (-$\sqrt{3}$, 9) is a relative maximum

At x = $\sqrt{3}$, f"($\sqrt{3}$) <0 so that ($\sqrt{3}$ ,9) is a relative maximum

# 2.3 Graph Sketching: Asymptotes and Rational Functions.

A rational function is the ratio of two polynomials

Examples 1) $f(x) = \dfrac{2x+4}{x+1}$

2) $f(x) = \dfrac{x+1}{x^2+3}$

3) $g(x) = \dfrac{4x^2+5}{x^2+5x+6}$

4) $h(t) = \dfrac{5t^3+t^2+2t}{t^2-3t+2}$

Vertical Asymptotes correspond to points x where there is a division by 0

Example: 1) A vertical asympote for the curve y = f(x) is $x=-1$

2) No vertical asymptote because $x^2+3$ is never 0

3) $x^2+5x+6 = (x+2)(x+3)$ $x=-2$ and $x=-3$ are the two vertical asymptotes

4) $t^2-3t+2 = (t-2)(t-1)$ $t=1$ and $t=2$ are vertical asymptotes

Horizontal Asymptote to $y=f(x)$ correspond to a straight line y = constant so that the curve gets close to the line for large x (positive or negative)

For rational functions factor out the highest power downstairs

$$\lim_{x\to\infty} \frac{1}{x} = 0 \qquad \lim_{x\to-\infty} \frac{1}{x} = 0$$

Example 1) $f(x) = \dfrac{2x+4}{x+1} = \dfrac{\cancel{x}}{\cancel{x}}\dfrac{(2+4/x)}{(1+1/x)}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2+4/x}{1+1/x} = \frac{2+0}{1+0} = 2$$

Horizontal Asymptote is $y = 2$ at $x = \infty$ and at $x = -\infty$

So y=2 is a horizontal asymptote.

Taking the limit as x goes to $-\infty$ gives the same value y=2 is a horizontal asymptote at +/- $\infty$

Example 2) 2) $f(x) = \dfrac{x+1}{x^2+3}$ (degree on bottom larger)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x+1}{x^2+3} = \lim_{x\to\infty} \frac{\cancel{x^2}\left(1/x + 1/x^2\right)}{\cancel{x^2}\left(1 + 3/x^2\right)} = \lim_{x\to\infty} \frac{\overset{\to 0}{1/x} + \overset{\to 0}{1/x^2}}{1 + \underset{\to 0}{3/x^2}} = \frac{0}{1} = 0$$

So the horizontal asymptote is y = 0

(Same as x approaches $-\infty$ )

Example 3) $g(x) = \dfrac{4x^2+5}{x^2+5x+6}$ (degree on bottom equals degree on top)

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{\cancel{x^2}}{\cancel{x^2}} \frac{4+5/x^2}{1+5/x+6/x^2} = \lim_{x\to\infty} \frac{4+5/x^2}{1+5/x+6/x^2} = \frac{4+0}{1+0+0} = 4$$

So y = 4 is a horizontal asymptote

Example 4) $h(t) = \dfrac{5t^3 + t^2 + 2t}{t^2-3t+2}$ (degree on bottom is one smaller than one on top

$$h(t) = \frac{\cancel{t^2}}{\cancel{t^2}} \frac{5t+1+2/t}{1-3/t+2/t^2} = \frac{5t+1+\overset{\to 0}{2/t}}{1-\underset{\to 0}{3/t}+\underset{\to 0}{2/t^2}} \approx \frac{5t+1+0}{1-0}$$

So y = 5t+1 is an oblique asymptote.

If the degree on top is 2 or more larger than that on the bottom then this doesn't work. There is no asymptote

Example $y = \dfrac{x^3}{x + 1}$ does not have a horizontal or oblique asymptote

Example: Graph $y = \dfrac{x}{x - 2}$

Solution a) intercepts

b) Asymptotes (and domain)