For $f(x) = x^2+3x-4$ compute $\frac{f(x+h)-f(x)}{h}$.

and then compute $f'(x)$

$f(x+h) = (x+h)^2 + 3(x+h) - 4 = \cancel{x^2} + 2xh + h^2 \cancel{+3x} + 3h \cancel{-4}$

$f(x) = \cancel{x^2} \cancel{+3x} \cancel{-4}$

$f(x+h)-f(x) = 2xh + h^2 + 3h = h(2x+h+3)$

$$\frac{f(x+h)-f(x)}{h} = \frac{\cancel{h}(2x+h+3)}{\cancel{h}} = 2x+h+3$$

Therefore $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim 2x + h + 3 = 2x+3$

Review from Section 1.2

Definition: A function $f(x)$ is continuous at $x = a$ if

1) $f(a)$ exists (that is a is in the domain of f)
2) $\lim_{x \to a} f(x)$ exists
3) $\lim_{x \to a} f(x) = f(a)$

The function $f(x)$ is continuous if it is continuous at every point a in its domain

Graphicly, $f(x)$ is continuous if you can draw the graph without removing pen from paper.

Example: If

$$f(x) = \begin{cases} x^2 \text{ if } x<2 \\ x+b \text{ if } x \geq 2 \end{cases}$$

then

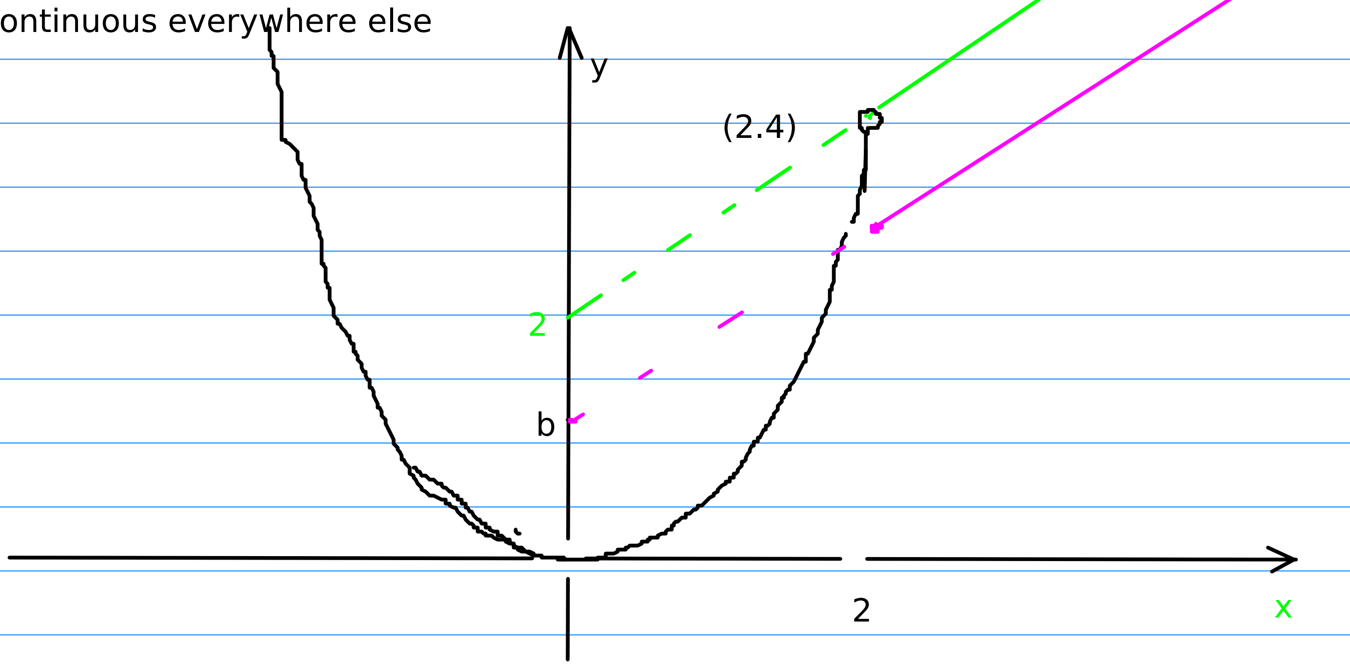
$\lim_{x \to 2^-} f(x) = 4$ $\quad$ (= lim $x^2=4$)

$\lim_{x \to 2^+} f(x) = 2+b = f(2)$

y = x+2

For continuity at $x=2$, need $2+b=4$

So f is continuous at $x=2$ if and only if a =2 and it is continuous everywhere else



Recall from Wed. June 19

Power Rule $\frac{d}{dx} x^n = nx^{n-1}$

all n = 0,1,2,3, (and in fact all n)

Example $\frac{d}{dx} x^7 = 7x^6$ n = 7

Example $\frac{d}{dx} \sqrt{x} = \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$ n = 1/2

as we saw in an example in Section 1.4.

Sum-Difference Rule $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$

Theorem 3 If c is a constant $\frac{d}{dx} cf(x) = c \frac{d}{dx} f(x)$

These rules follow directly from the corresponding rules for limits.

Example; Differentiate y = 3x^5+ 4x^2+2x+11

y' = 3(5x^4) + 4(2x) + 2 + 0 = 15x^4 + 8 x + 2

11=11x^0

Example If $y = x^2 + \frac{1}{x^2}$ then what is y'?

y = x^2 + x ^(-2)

y' = 2x +(-2 x^(-3)) = $2x - \frac{2}{x^3}$

Monday June 24

1.6 Differentiation Techniques: The Product and Quotient Rules.

Product Rule: If F(x) =f(x)g(x) (and f and g are differentiable at x) then

$$F'(x) = \frac{d}{dx}\left(f(x)\, g(x) \right) = f(x)\left[\frac{d}{dx} g(x)\right] + g(x)\left[\frac{d}{dx} f(x)\right]$$

(Alternative notation: F'(x) = f(x)g'(x)+g(x)f'(x).)

Example: Differentiate (x^3+ 4x +7)(x^2+5x) using the product rule.

$$\frac{d}{dx}(x^3+ 4x +7)(x^2+5x) = (x^3+4x+7)(2x+5) + (x^2+5x)(3x^2+4)$$

$$\frac{d}{dx}\, x^3+4x+7 = 3x^2 +4$$

Example: Differentiate $y = x^3\sqrt{x}$

Recall that $\frac{d}{dx}\sqrt{x} = \frac{d}{dx}\, x^{(1/2)} = \frac{1}{2}\, x^{(-1/2)}$ (power rule)

so that, by the product rule

$$y' = x^3\left(\frac{1}{2}\, x^{(-1/2)}\right) + x^{(1/2)}(3x^2)$$

$$= \frac{1}{2}\, x^{(5/2)} + 3\, x^{(5/2)} = \frac{7}{2}\, x^{(5/2)}$$

This agrees with the power rule if we observe that $x^3\sqrt{x} = x^{7/2}$

$= x^3\, x^{(1/2)}$

Quotient Rule: If $Q(x) = \frac{N(x)}{D(x)}$ then

$$Q'(x) = \frac{D(x)\,N'(x) - N(x)D'(x)}{[D(x)]^2}$$

(Assuming N and D are differentiable and $D(x) \neq 0$.)

Example: Differentiate $Q(x) = \frac{7x}{x^2 - 4x + 1}$

$$Q'(x) = \frac{(x^2-4x+1)7 - (7x)(2x-4)}{[x^2-4x+1]^2} = \frac{\cancel{7x^2} - \cancel{28x} + 7 - \overset{-7}{\cancel{14x^2}} + \cancel{28x}}{[x^2-4x+1]^2}$$

$$= \frac{7-7x}{[x^2-4x+1]^2}$$

Example: Differentiate $Q(x) = \frac{1}{x^3}$

$$Q'(x) = \frac{x^3 \cdot 0 - 1(3x^2)}{[x^3]^2} = \frac{-3x^2}{x^6} = -3\,\frac{1}{x^4} = -3x^{(-4)}$$

Alternatively $Q(x) = x^{-3}$ Power rule says $Q' = -3x^{-4}$

Example

$$Q(x) = \frac{x^3+4x}{x^2+5}$$

$$Q'(x) = \frac{(x^2+5)(3x^2+4) - (x^3+4x)(2x)}{[x^2+5]^2}$$

$$= \frac{3x^4 + 4x^2 + 15x^2 + 20 - 2x^3 - 8x^2}{[x^2+5]^2}$$

$$= \frac{3x^4 - 2x^3 + 11x^2 + 20}{[x^2+5]^2}$$

1.7 Chain Rule

Extended Power Rule: $\frac{d}{dx}[f(x)]^k = k\,[f(x)]^{k-1}\,\frac{df}{dx}(x)$

Example Differentiate y = (x^2+2)^7 f(x) =x^2+2 k=7

$y' = 7(x^2+2)^6\,2x = 14x(x^2+2)^6$

Example Let $f(x) = \sqrt{4x + 11}$ = (4x+11)^(1/2)

Find

$f'(x) = \frac{1}{2}$ (4x+11)^(-1/2)*4= 2(4x+11)^(-1/2) $= \frac{2}{\sqrt{4x+11}}$

Example: Let $g(x) = x\sqrt{4x + 11}$ Find g'(x).

Use the product rule.

$g'(x) = \sqrt{4x+11} + x\,\frac{2}{\sqrt{4x+11}}$

Composition of Functions: Given two functions f and g it is possible to form a new function

$f \circ g(x) = f(g(x))$

For example $f(x) = \sqrt{x}$ g(x) = 1/x $f(g(x)) = \sqrt{1/x}$ = $1/\sqrt{x}$

(Press the reciprocal

button and then the square root button on your calculator.)

Example: Express F(x) = (x^3+5x)^(1/3) as f(g(x)). That is find f and g.

g(x) = x^3+5x f(x) = x^(1/3) Then F(x) = f(g(x))

Example If F(x) = $\dfrac{4}{x^2+4}$ then express F(x) as f(g(x))

g(x) = x^2+4 f(x) = 4/x then F(x) = f(g(x))

Chain Rule:

$f \circ g' = f'(g(x))g'(x)$

$f \circ g'(x) = f'(g(x))g'(x)$

Example: Differentiate F(x) = (x^3+5x)^(1/3) g(x) =x^3+5x f(x) = x ^(1/3)

F'(x) = (1/3)(x^3+5x)^(-2/3) (3x^2+5)

f'(x) = 1/3x^(-2/3) f'(g(x)) = 1/3 (x^3+5x)$^{(-2/3)}$

Example: Differentiate F(x) = $\dfrac{4}{x^2+4}$ = 4 (x^2+4)^(-1)

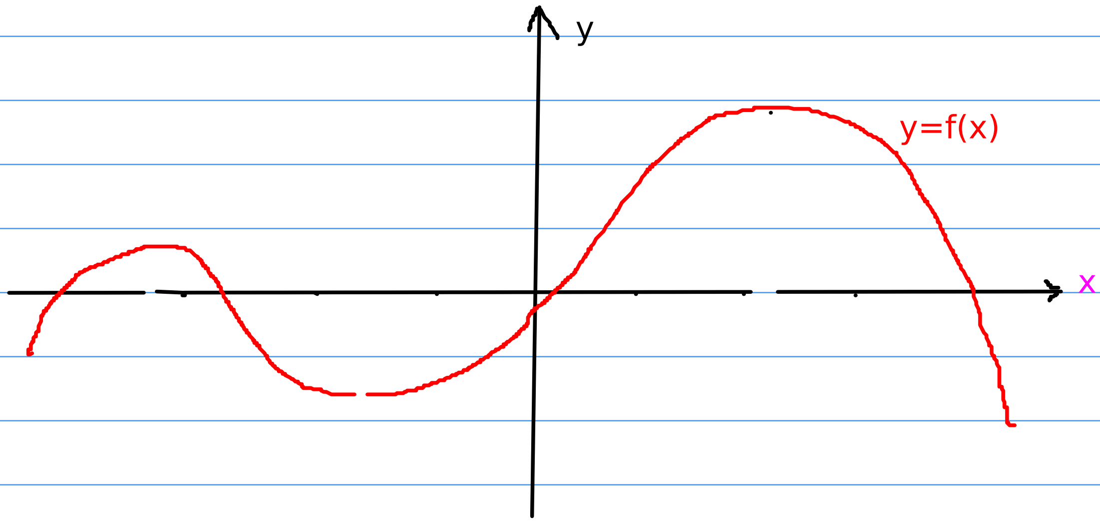
F'(x) = 4 (-1) (x^2+4)^(-2) (2x)

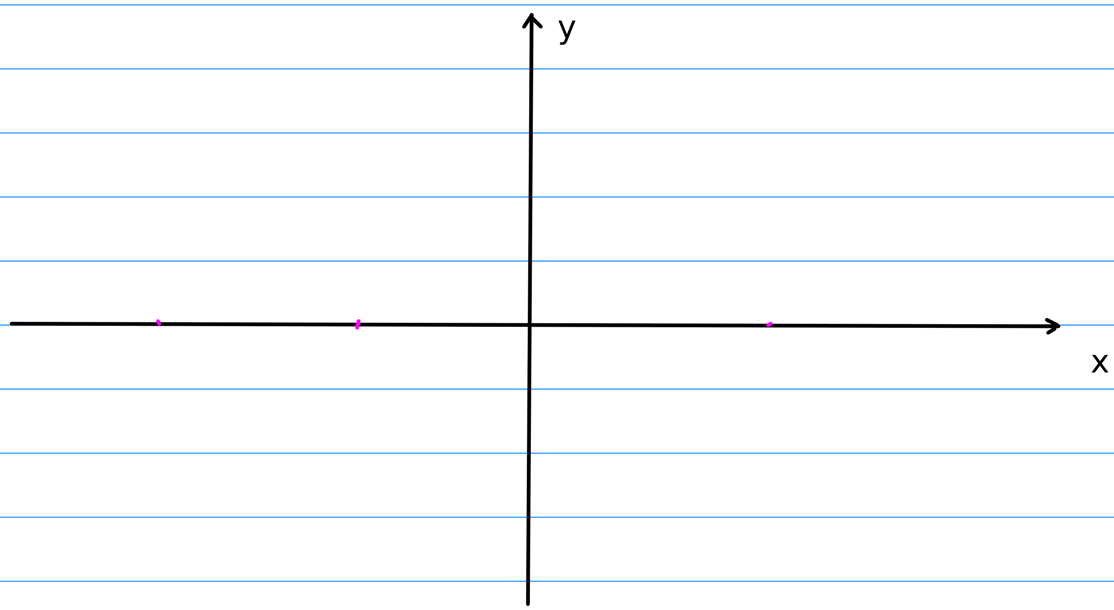
=-8 x (x^2+4)^(-2)

= $\dfrac{-8x}{(x^2+4)^2}$

1.8 Higher Order Derivatives.

The Derivative as a Function





Example: If $f(x) = x^3 + 4x + 8$ then

the first derivative is $f'(x) = \frac{d}{dx} f =$ and

the second derivative is $f''(x) = \frac{d^2}{dx^2} f(x) =$

the third derivative is $f'''(x) = \frac{d^3}{dx^3} f(x) =$

If s(t) is a function of time that indicates where along a straight line an object is then

$$s'(t) = \lim_{h \to 0} \underbrace{\frac{s(t+h) - s(t)}{h}}$$

displacement over the time interval [t,t+h] divided by time elapsed (that is average velocity)

s'(t) is the (instantaneous) velocity.

s''(t) is the

Example A ball thrown from 6 feet off the ground with initial velocity 64 ft/sec is

$s(t)=-16t^2 + 64 t +6$ feet

above ground level t seconds later.

$v(t) = s'(t) =-32t + 64$

so that the ball continues up for 2 seconds and then falls.

$a(t) =s''(t)= -32$ is the acceleration due to gravity. (in feet per second^2)

Summary: First derivative of position (or displacement) is velocity and the second derivative is acceleration.

2.1 Using the first Derivative to Find Maximum and Minimum and Sketch Graphs

decreasing

A function f is said to be increasing on an interval I if for any $a<b$, a,b in I

$f(a)<f(b)$

$f(a)>f(b)$

If $f(x)$ is differentiable on I and if $f'(a)>0$ then

$$\frac{f(a+h)-f(a)}{h} > 0 \qquad \text{(for } h>0 \text{ or } h<0\text{)}$$

so f is increasing.

Theorem 1 If $f'(x)>0$ for all x in $I = (c,d)$ then f is increasing on I

If $f'(x)<0$ for all x in $I = (c,d)$ then f is decreasing on I

Definition: A number c is a critical number of a function $f(x)$ if either

$f'(c) = 0$

or f' does not exist at c

Example: $f(x) = x^2-6x + 3$ has a critical point at ?

$f'(x) = 2x-6$

Set to 0 $2x-6 = 0$ so that $x=3$ is the only critical point of f.

$$\text{Example } f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Critical points of f?

The critical points are the points where f can change from being increasing to being decreasing or the reverse (decreasing to increasing)