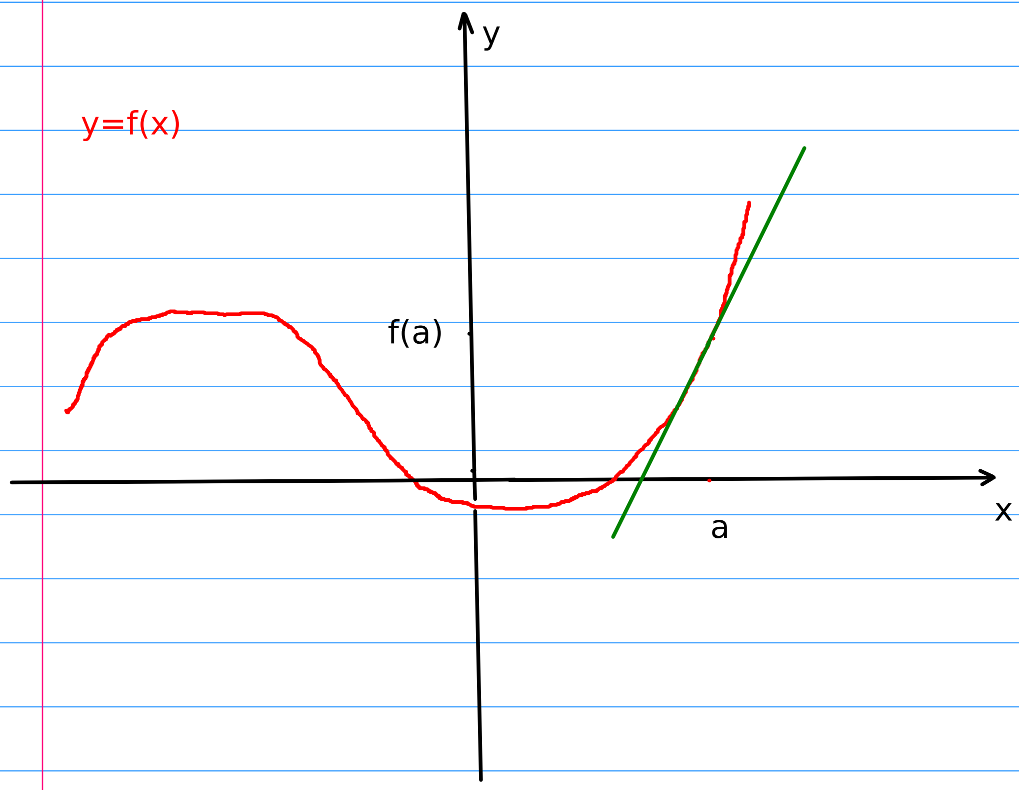
Recall

from 6/12

1.1 Limits: A Numerical and Graphical Approach.

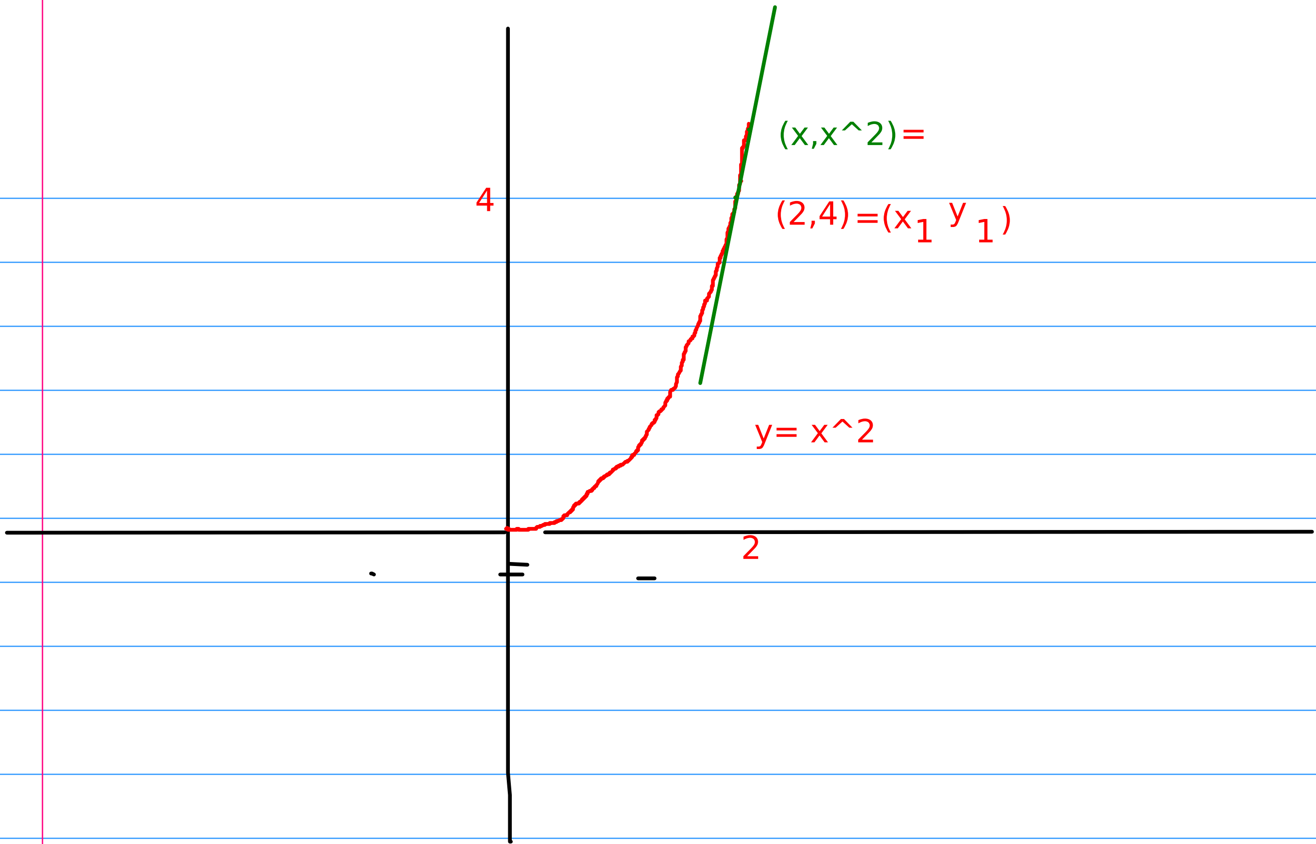
Orientation: We begin our study of calculus. We shall introduce ``differential calculus'' by talking about the slope of the graph of a function. We already know about the slope of lines and this is a generalization to curves. The slope of a graph y = f(x) at a point (a,f(a)) on the graph is the slope of the tangent line. What is the tangent line?



Example: f(x) = x^2 and a = 2. The tangent line at (2,4) must pass through (2,4) but we do not know another point on the line nor the slope. We approximate. Suppose $x \neq 2$ but x is near 2. Then (x,x^2) is on the curve and near (2,4). The slope of the ``secant'' line through (2,4) and (x,x^2) is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{x^2-4}{x-2}$$

x is near 2 or in the limit x is 2



limit as x approaches 2 of $\frac{x^2-4}{x-2}$ is, we shall see 4

The slope of the tangent line is 4

Try x =2.1 x^2 = 4.41

$\frac{4.41-4}{2.1-2} = 4.1$

We shall see that $\lim_{x \to 2} \frac{x^2-4}{x-2} = 4$

which means that if we plug in values of x closer and closer to x=2 like x =2.001 or x = 1.99999 then (x^2-4)/(x-2) gets closer and closer to 4.

Recall from Monday June 17

1.3 Average Rates of Change:

Example: $\lim_{h\to 0} \frac{(1+h)^3 - 1}{h} = \lim_{h\to 0} \frac{1}{h}\left(\cancel{1}+3h + 3h^2 + h^3 \cancel{-1}\right)$

y = x^3 at (1,1)

Secant line through (1,1) and (1+h,(1+h)^3)

$= \lim \frac{1}{\cancel{h}} \cancel{h}(3+3h + h^2)$

$= \lim\ 3 + 3h + h^2 = 3$

Example: $\lim_{h\to 0} \frac{\frac{2}{3+h} - \frac{2}{3}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{2(3) - 2(3+h)}{3(3+h)}\right)$

$= \lim \frac{1}{h} \frac{\cancel{6}-\cancel{6}-2h}{3(3+h)}$

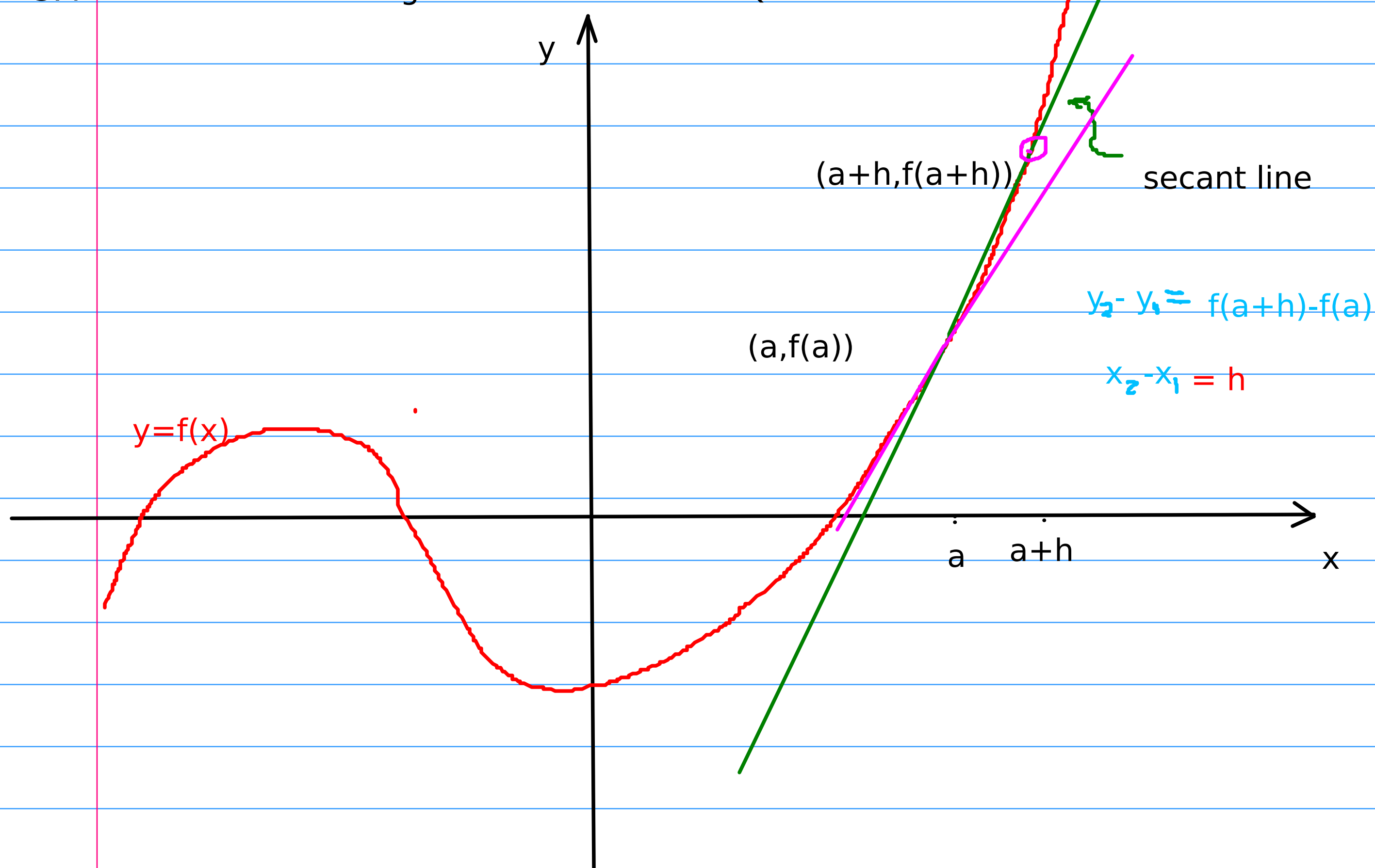
$= \lim \frac{1}{\cancel{h}} \frac{-2\cancel{h}}{3(3+h)} = \frac{-2}{9}$

Example: $\lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} = \lim \frac{1}{h} \frac{\cancel{4}+h - \cancel{4}}{\sqrt{4+h} + 2}$

$= \lim \frac{1}{\sqrt{4+h} + 2} = \frac{1}{4}$

Wednesday, June 19

3.4 Differentiation Using Limits of Difference Quotients



A secant line cuts the curve at a second place near the point of interest (a,f(a)).

In the picture the second point is (a+h,f(a+h)) where we are thinking that h is small and not necessarily positive. The slope of the secant line is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$$

and we refer to this last expression as a difference quotient for f at a.

Definition: The derivative of f at a is denoted f'(a) and is defined to be

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

provided the limit exists. If the limit does not exist then f is not differentiable at a

Leibniz Notation: $f'(a) = \frac{df}{dx}(a) = f'(a)$

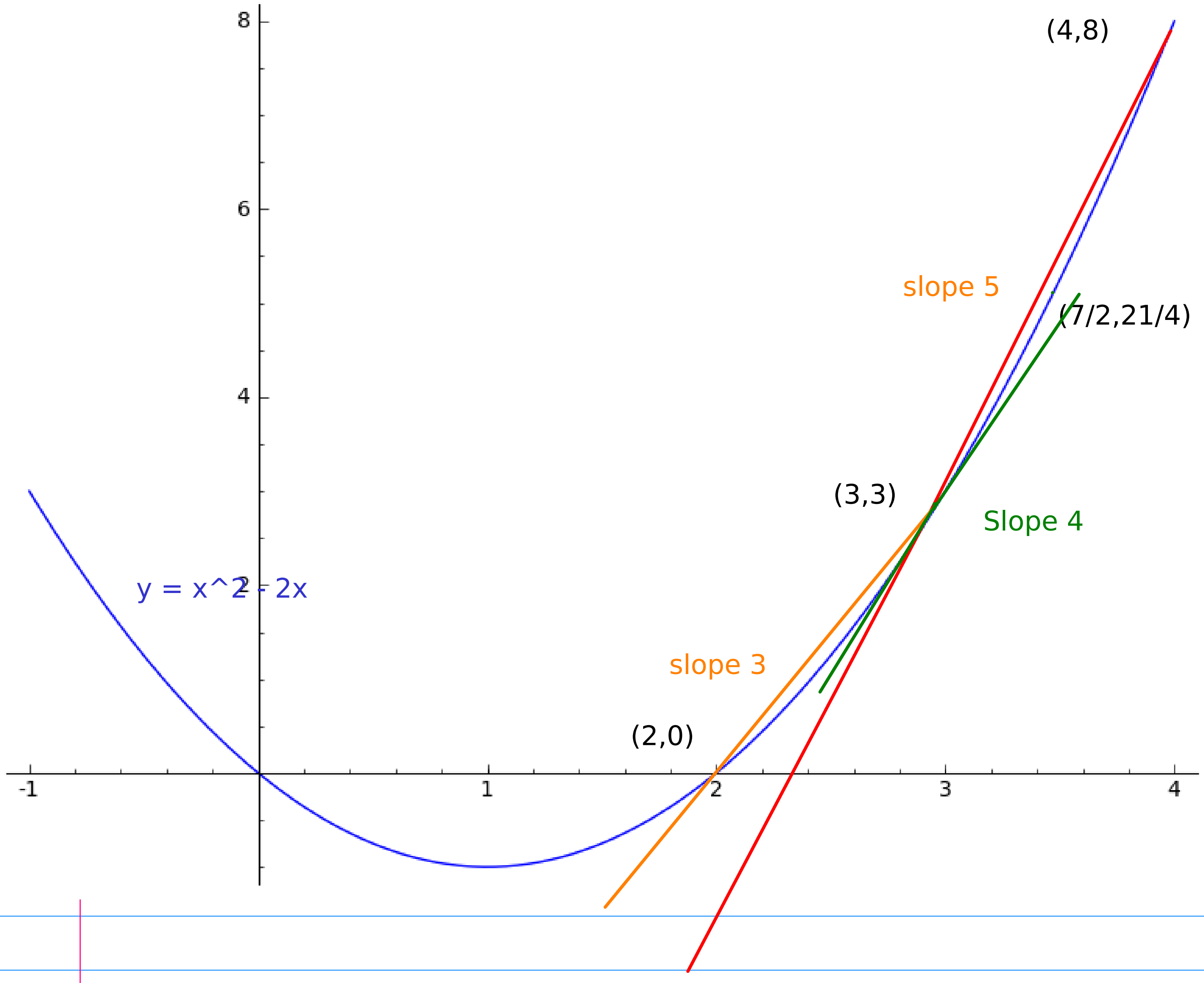
Physical interpretation: f'(a) is the slope of the tangent line because it is the limit of the slope of secant lines through points on the graph near (a,f(a))

Example: Find the slope of the tangent line of $y = f(x) = x^2 - 2x$ at $a = 3$

First the slope of the secant lines through (3,3) and say (4, 4^2-2(4))= (4,8)

Red line has slope $\frac{8-3}{4-3} = 5;$

Orange line has slope $\frac{3-0}{3-2} = 3$



What about the secant line that passes through (7/2,21/4) (Note if x =7/2 which is reasonably near 3 then y = x^2-2x= 21/4.)
the slope is

$$\frac{21/4-3}{7/2-3} = 9/2$$

What is the slope of the tangent line?

$$\lim_{h \to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{(3+h)^2-2(3+h) - 3}{h}$$

$$\frac{(3,3)\quad(3+h,(3+h)^2-2(3+h))}{}$$

$$= \lim_{h\to 0} \frac{\cancel{9} +6h+h^2-\cancel{6}-2h - \cancel{3}}{h}$$

$$= \lim_{h\to 0} \frac{4h +h^2}{h} \qquad \frac{\cancel{h}(4+h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 4+h$$

It is 4 !!

Note 3<4<9/2 as the picture suggests.

Observe that once we know the slope of the tangent line then we know an equation of the tangent line. It is

Read Example 3 on page 137. Observe that f(x)=3x-4 has derivative f'(x) =3.
In general the slope of any tangent line to f(x) = mx + b is m as it should be.
We have generalized the notion of slope from a line to a curve.

Example: Find f'(4) if f(x) =$\sqrt{x}$. Then find f'(a) for any a>0.

$$f'(4) = \lim \frac{f(4+h) - f(4)}{h}$$

$$f'(4) = \lim_{h\to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} = \lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2}$$

$$= \lim_{h\to 0} \frac{\cancel{4}+h -\cancel{4}}{h(\sqrt{4+h} +2)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}^{\,1}}{\cancel{h}\left(\sqrt{4+h}+2\right)}$$

$$= \frac{1}{\sqrt{4+0}+2} = 1/4$$

$$f'(a) = \lim_{h \to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} = \lim_{h \to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} \cdot \frac{\sqrt{a+h} + \sqrt{a}}{\sqrt{a+h} + \sqrt{a}}$$

$$= \lim_{h \to 0} \frac{\cancel{a}+\cancel{h}^{\,1} - \cancel{a}}{\cancel{h}\left(\sqrt{a+h} + \sqrt{a}\right)}$$

$$= \lim \frac{\cancel{h}}{\cancel{h}\left(\sqrt{a+h} + \sqrt{a}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{a+h} + \sqrt{a}}$$

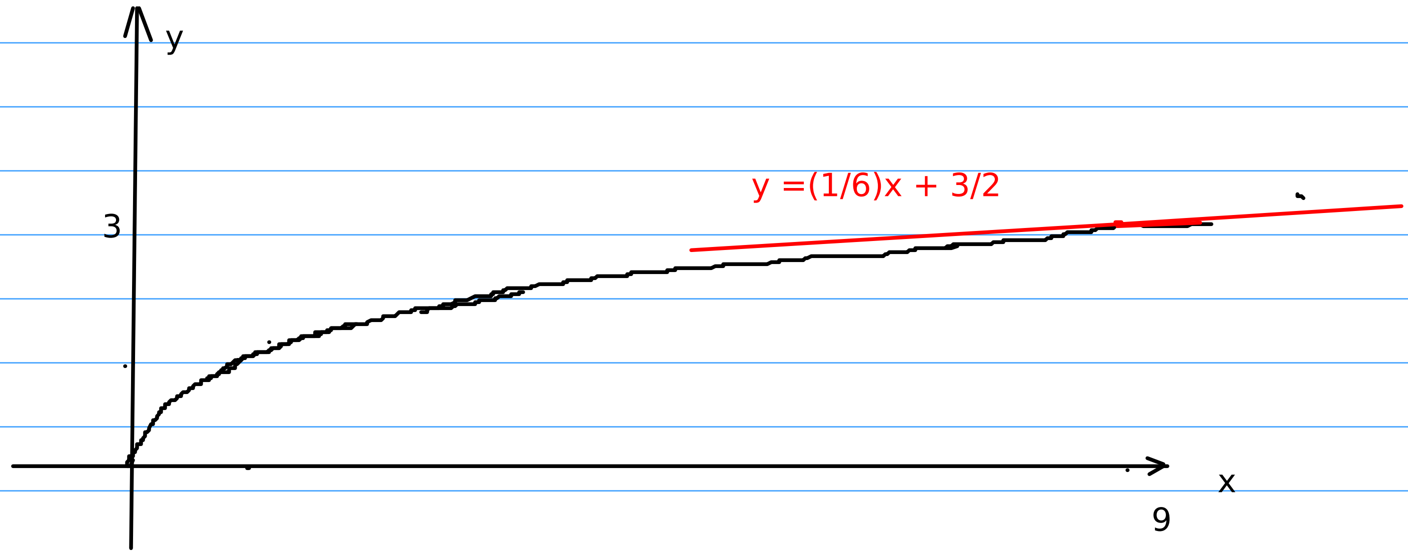
$$= \frac{1}{\sqrt{a} + \sqrt{a}}$$

So $f'(a) = \dfrac{1}{2\sqrt{a}}$ or $f'(x) = \dfrac{1}{2\sqrt{x}}$ if $x>0$

For example if $x = 9$ the slope of the tangent line is 1/6 and the equation is

$y - 3 = (1/6)(x-9)$ (because when $x = 9$ $y = \sqrt{9} = 3$)

$y = (1/6)x + 3/2$



Example (Non Differentiable)

Find the derivative of $f(x) = |x|$ at $x = 0$ if it exists.

We need to check the existence of

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|-0}{h} = \lim \frac{|h|}{h}$$

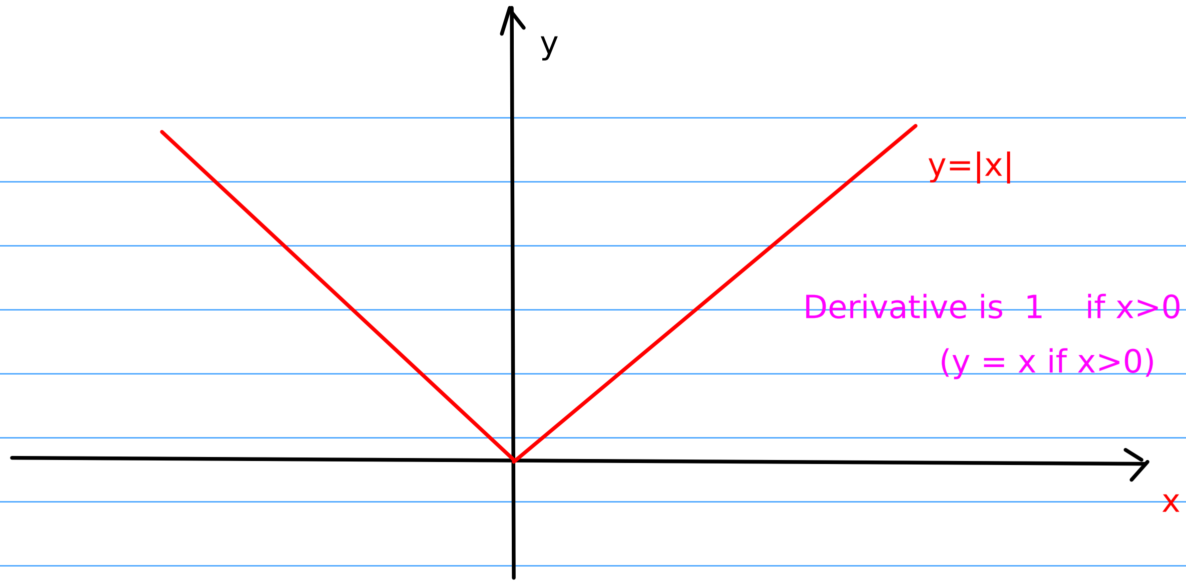
Recall that $|h| = h$ if $h>0$ and $|h|=-h$ if $h<0$

So $$\lim_{h\to 0+} \frac{|h|}{h} = \lim_{h\to 0+} \frac{h}{h} = \lim_{h\to 0+} 1 = 1$$

and $$\lim_{h\to 0-} \frac{|h|}{h} = \lim_{h\to 0-} \frac{-h}{h} = \lim_{h\to 0-} -1 = -1$$

The limit does not exist

$f(x) = |x|$ is not differentiable at $x = 0$



No tangent line at (0,0). Everywhere else |x| is differentiable and the derivative is 1 if x>0 and is -1 if x<0.

1.5 Differentiation Techniques: The Power and Sum-Difference Rules

Now we try and make finding f'(x) for simple functions f(x) easier and faster.

Example: If f(x) = x then f'(x) = 1

Example: If f(x) = x^2 then $f'(x) = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh + h^2 - \cancel{x^2}}{h}$

$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$

$= \lim_{h\to 0} 2x+h$

$=2x$

Example: If f(x) = x^3 then $f'(x) = \lim_{h\to 0} \frac{(x+h)^3-x^3}{h}$

$= \lim_{h\to 0} \frac{\cancel{x^3} +3x^2h +3xh^2 + h^3-\cancel{x^3}}{h}$

$= \lim_{h\to 0} \frac{\cancel{h}(3x^2 +3xh+h^2)}{\cancel{h}}$

$= \lim_{h\to 0} 3x^2+3xh + h^2 = 3x^2$

Example If f(x) = x^n then $f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$

$(x+h)^n =(x+h)(x+h) \ldots (x+h)$

$= \lim_{h\to 0} \frac{\cancel{x^n} + nx^{(n-1)}h +h^2(\ldots) -\cancel{x^n}}{h}$

$= \lim_{h\to 0} \underline{n x^{(n-1)}} +h(\ldots)$

$=nx^{(n-1)}$

Power Rule $\frac{d}{dx} x^n = nx^{n-1}$

all n = 0,1,2,3, (and in fact all n)

Example $\frac{d}{dx} x^7 = 7x^6$ n = 7

Example $\frac{d}{dx}\sqrt{x} = \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

as we saw in an example in Section 1.4.

Sum-Difference Rule $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$

Theorem 3 If c is a constant $\frac{d}{dx} cf(x) = c \frac{d}{dx} f(x)$

These rules follow directly from the corresponding rules for limits.

Example; Differentiate y = 3x^5+ 4x^2+2x+11

y' = 3(5x^4) + 4(2x) + 2 + 0 = 15x^4 + 8 x + 2

11=11x^0

Example If $y = x^2 + \frac{1}{x^2}$ then what is y'?

y = x^2 + x ^(-2)

$y' = 2x + (-2\, x^{-3}) = 2x - \frac{2}{x^3}$