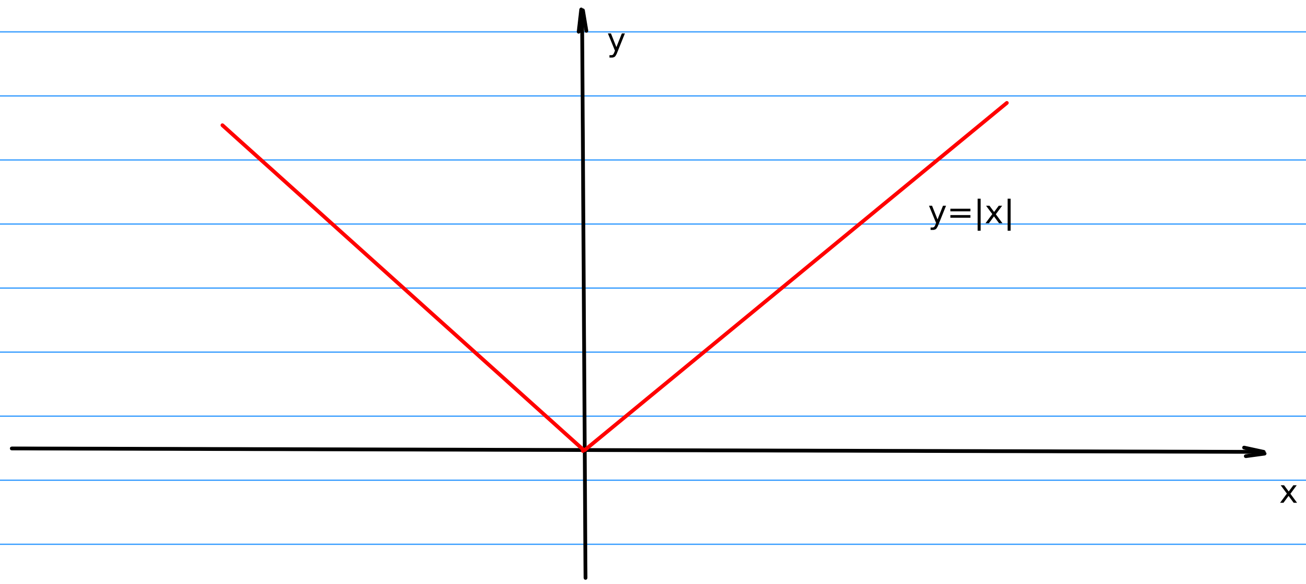
Math 1730, Monday June 17

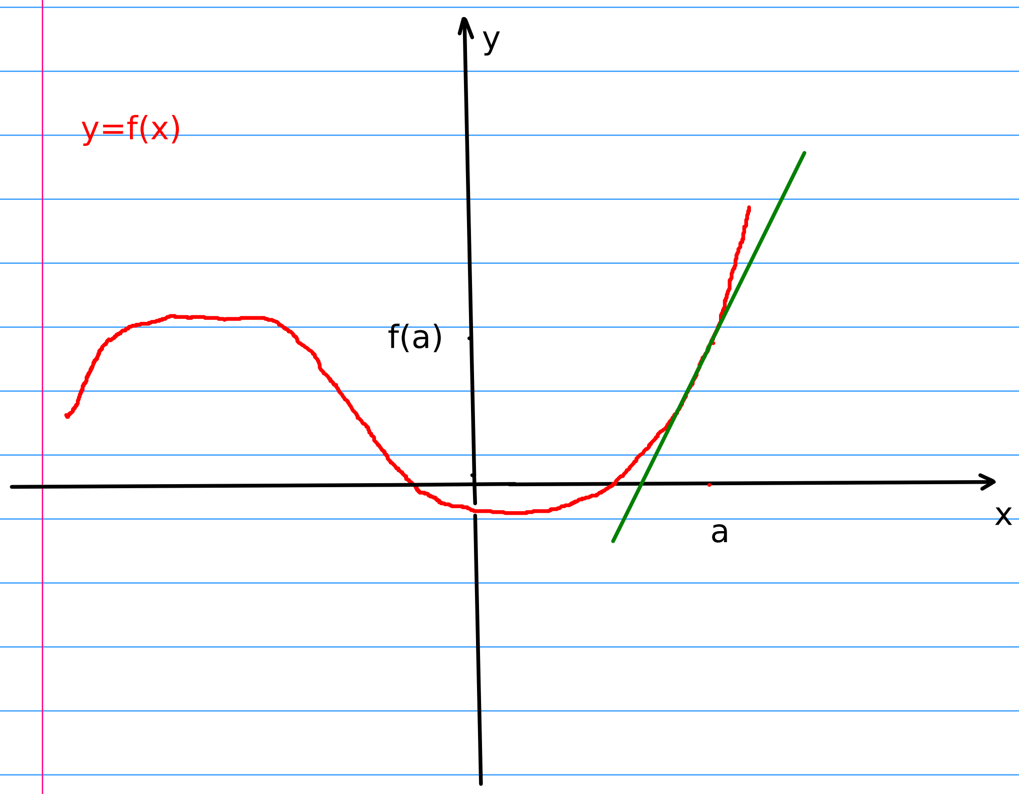
Don't miss the absolute value function $f(x) = |x|$ of Example 8, page 60.



Recall from 6/12

1.1 Limits: A Numerical and Graphical Approach.

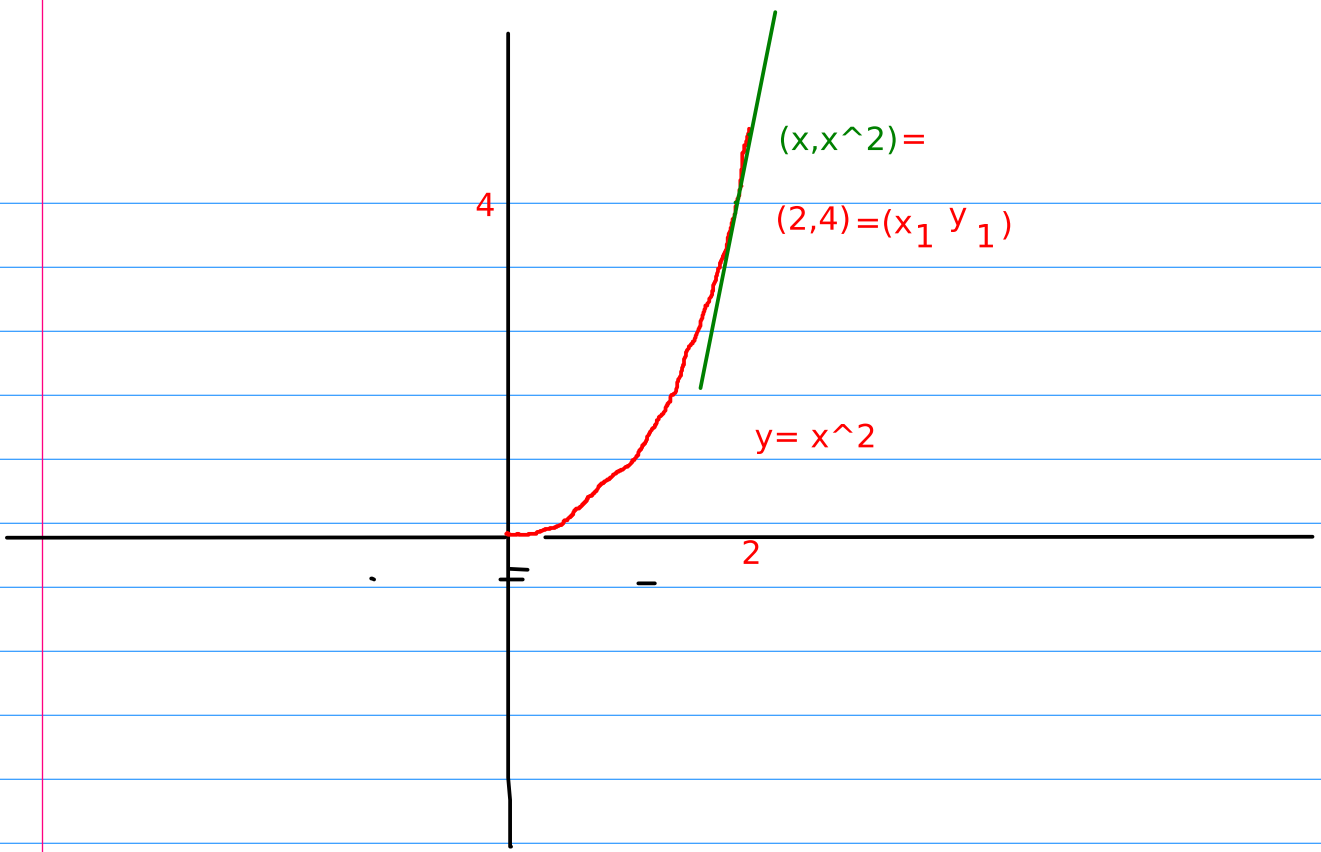
Orientation: We begin our study of calculus. We shall introduce ``differential calculus'' by talking about the slope of the graph of a function. We already know about the slope of lines and this is a generalization to curves. The slope of a graph $y = f(x)$ at a point $(a,f(a))$ on the graph is the slope of the tangent line. What is the tangent line?



Example: $f(x) = x^2$ and $a = 2$. The tangent line at $(2,4)$ must pass through $(2,4)$ but we do not know another point on the line nor the slope. We approximate. Suppose $x \neq 2$ but x is near 2. Then (x,x^2) is on the curve and near $(2,4)$. The slope of the ``secant'' line through $(2,4)$ and (x,x^2) is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{x^2-4}{x-2}$$

x is near 2 or in the limit x is 2



limit as x approaches 2 of $\dfrac{x^2-4}{x-2}$ is, we shall see 4

The slope of the tangent line is 4

Try x =2.1 x^2 = 4.41

$$\frac{4.41-4}{2.1-2} = 4.1$$

We shall see that $\lim_{x \to 2} \dfrac{x^2-4}{x-2} = 4$

which means that if we plug in values of x closer and closer to x=2 like x =2.001 or x = 1.99999 then (x^2-4)/(x-2) gets closer and closer to 4.

In Section 1.1 we are therefore considering $\lim_{x \to a} F(x)$

and the case of interest is when F(a) may not make sense (like 0/0).

Definition: We say that the limit of f(x) as x approaches a is L and write

$$\lim_{x \to a} f(x) = L$$

if $|f(x)-L|$ can be made arbitrarily small by choosing $|x-a|$ small but $x \neq a$.

Notice that f(a) plays no role in the definition ($x \neq a$) Nevertheless if

f is a ``nice'' function near a then

$$\lim_{x \to a} f(x) = f(a)$$

and so the limit concept extends evaluation from nice functions to not-so-nice functions. However the limit may not exist for some

nastier functions.

Example: Limits from Graphs (Annotated pdf file ... see website)

Example:

$$\lim_{x \to 2} x^2 = 4$$

$x^2-4 = (x-2)(x+2)$

because $|x^2-4| = |x-2||x+2|$ and so by making $|x-2|$ small we can force $|x^2-4|$ to be small too.

In general

$$\lim_{x \to a} x^2 = a^2$$

1.2 Algebraic Limits and Continuity:

Example: Find $\displaystyle\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$ $\quad\left(= \frac{0}{0} \right)$

Recall that

$$\frac{x^2 - 4}{x - 2} = \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = x + 2 \text{ when } x \neq 2$$

The definition of limit given before specificly excludes x = 2 from consideration and so we can write

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} x + 2 = 4$$

(Is it clear that as x gets close to 2, x + 2 gets close to 4?)

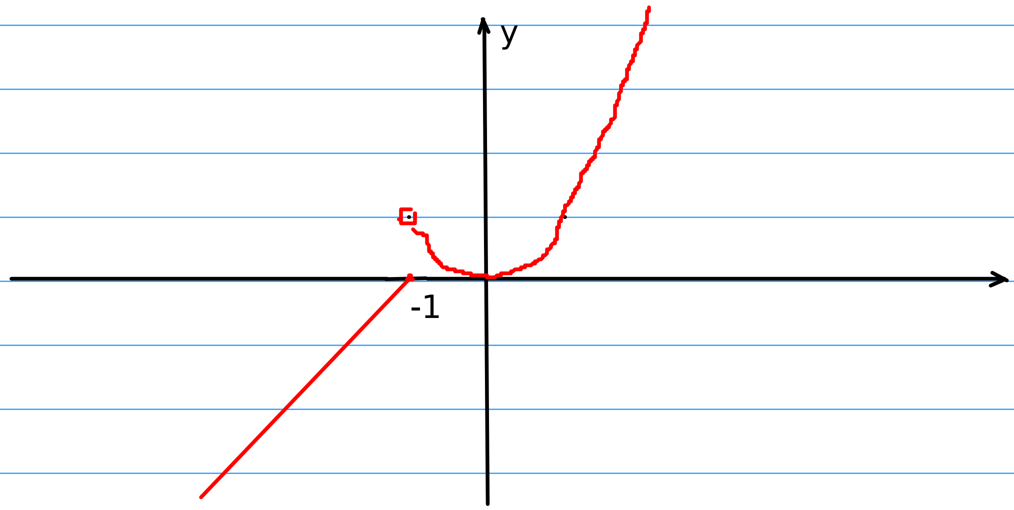
Example: Evaluate $\displaystyle\lim_{x \to -1} \frac{x^2+4x+3}{x^2 - 1} = \lim_{x \to -1} \frac{\cancel{(x+1)}(x+3)}{\cancel{(x+1)}(x-1)}$

$$= \underline{-1}$$

Example Let

$$f(x) = \begin{cases} x^2 & \text{if } x > -1 \\ x+1 & \text{if } x \leqslant -1 \end{cases}$$

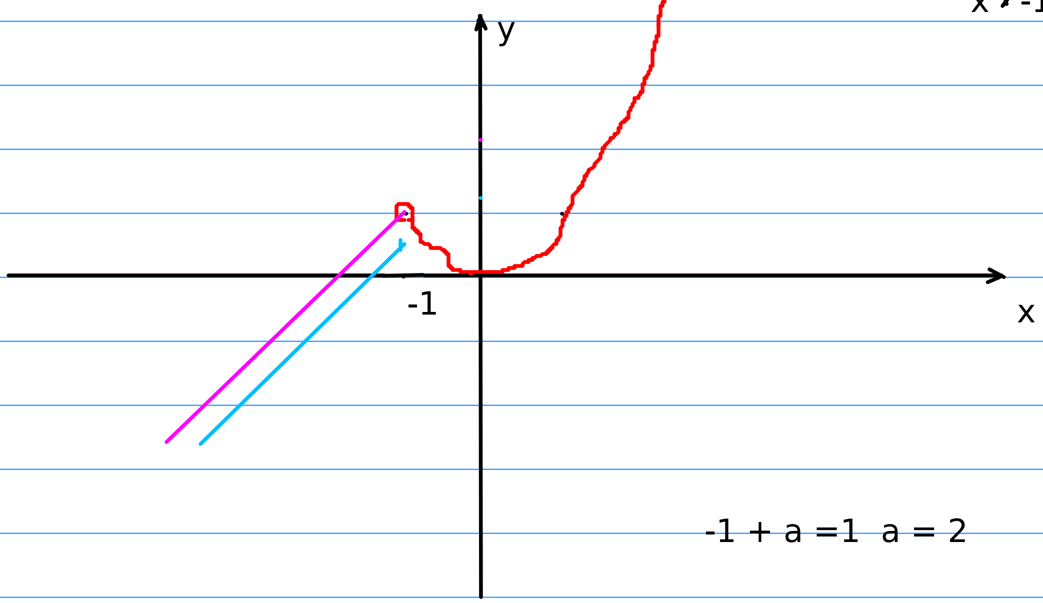
Find a) $f(-1) = 0$ b) $\displaystyle\lim_{x \to -1^-} f(x) = 0$ c) $\displaystyle\lim_{x \to -1^+} f(x) = 1$ d) $\displaystyle\lim_{x \to -1} f(x) =$ DNE



Example Let

$$f(x) = \begin{cases} x^2 & \text{if } x > -1 \\ x+a & \text{if } x \leq -1 \end{cases}$$

Find $f(-1) = -1+a$ $\quad \lim_{x \to -1^-} f(x) = -1+a$ $\quad$ c) $\lim_{x \to -1^+} f(x) = 1$ $\quad \lim_{x \to -1} f(x)$



$-1 + a = 1 \quad a = 2$

$\lim_{x \to -1} f(x)$ exists if and only if $a = 2$

In this case we say $f(x)$ is continuous at $x = -1$

Rules

1) $$\lim_{x \to a} c = c$$ c is a constant

2) $$\lim_{x \to a} x^r = a^r$$ (except if r<0 and a = 0)

3) $$\lim_{x \to a} f(x) \pm g(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

4) $$\lim_{x \to a} [f(x) \cdot g(x)] = [\lim_{x \to a} f(x)] \cdot [\lim_{x \to a} g(x)]$$

5) $$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$ provided $\lim_{x \to a} g(x) \neq 0$

6) $$\lim_{x \to a} c\, f(x) = c \lim_{x \to a} f(x)$$

Example: $$\lim_{x \to a} x = a$$

Example: $$\lim_{x \to 3} \frac{x^2 + x - 8}{x + 3} = \frac{4}{6} = \frac{2}{3}$$

Example: $$\lim_{x \to -2} x\sqrt{x+3} = -2 (-2+3)^{\wedge}(1/2) = -2(1) = -2$$

Example: $$\lim_{h \to 0} \frac{(2+h)^2 - 4}{h} = \lim_{h \to 0} \frac{1}{h}(\cancel{4} + 2(2)h + h^2 \cancel{-4})$$

$$= \lim \frac{1}{h}(4h + h^2) = \lim_{h \to 0} 4 + h = 4$$

$\frac{(2+h)^2-4}{h}$ is the average rate of change of x^2 over the interval [2,2+h]

In general $\frac{f(b) - f(a)}{b - a}$ is the average rate of change of f(x) over the interval [a,b] (the change in f divided by the change in x).

$(1+h)(1+h)(1+h)=(1+h)(1+2h+h^2)= 1 + 2h + h^2 + h + 2h^2 + h^3$

$= 1 + 3h + 3h^2 + h^3$

Example: $\lim_{h\to 0} \frac{(1+h)^3 - 1}{h} = \lim_{h\to 0} \frac{1}{h}(\cancel{1}+3h+3h^2+h^3\cancel{-1})$

$= \lim \frac{1}{\cancel{h}} \cancel{h}(3+3h+h^2)$

$= \lim 3 + 3h + h^2 = 3$

Example: $\lim_{h\to 0} \frac{\frac{2}{3+h} - \frac{2}{3}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{2(3) - 2(3+h)}{3(3+h)}\right)$

$= \lim \frac{1}{h} \frac{\cancel{6}-\cancel{6}-2h}{3(3+h)}$

$= \lim \frac{1}{\cancel{h}} \frac{-2\cancel{h}}{3(3+h)} = \frac{-2}{9}$

Example: $\lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \lim \frac{1}{h} \frac{\cancel{4}+h \cancel{-4}}{\sqrt{4+h}+2}$

$= \lim \frac{1}{\sqrt{4+h}+2} = \frac{1}{4}$