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Meromorphic Continuation of Scattering Matrices: Long Range, Stark Case

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Long range quantum mechanical scattering in the presence of a constant electric field of strength $F > 0$ is discussed. It is shown that the scattering matrix, as a function of energy, has a meromorphic continuation to the entire complex plane as a bounded operator on $L(\mathbb{R}^{n-1})$ where n is the space dimension. There is a marked contrast between this result and the comparable result in the Schrödinger case ($F = 0$). The scattering matrix is constructed using two Hilbert space wave operators and time dependent modified wave operators both and the constructions are compared.

Keywords Analytic continuation; Quantum scattering; Schrödinger; Stark effect.

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1. Introduction

We shall study the scattering matrix and show that it has a meromorphic extension to the entire complex plane as a bounded operator in the following context. A non-relativistic quantum particle in $\mathbb{R}^n$ is scattered by a long range potential V in the presence of a constant electric field. If we suppose that the field acts in the direction $\mathbf{e}_1 = (1, 0, \dots, 0)$ of $\mathbb{R}^n$ and is of constant strength $F > 0$ and make convenient normalizations then the corresponding Hamiltonian can be written as

$$H_0 = -\Delta + Fx_1 \quad \text{and} \quad H = H_0 + V.$$

Here $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ is the Laplacian; H is regarded as a perturbation of H_0 and the scattering is by the potential V in the presence of the linear electric potential Fx_1. H and H_0 are referred to as "Stark" or "Stark-effect" operators in honor of Johannes Stark (1874–1957) who explored the effect of an electric field on the spectrum of hydrogen and discovered what is now known as the Stark Effect.

It is known, under the assumptions on V stated in the Hypotheses below, and if $F > 0$, that the scattering matrix $S(\lambda)$ exists for almost all real λ (the "free"

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energy) and defines a unitary operator on $L^2(\mathbb{R}^{n-1})$. We shall show that $S(\lambda)$ has a meromorphic continuation in λ to the entire complex plane as a bounded operator on $L^2(\mathbb{R}^{n-1})$. The continuation is of considerable physical interest because the poles are "resonances" of the system but what is perhaps of more mathematical interest is the contrast between this result and the corresponding result in the case $F = 0$; see the discussion after the statement of Theorem 2.

The following assumptions on V will be made throughout. Introduce the notations $\langle x\rangle = (1+|x|^2)^{1/2}$ and $x^- = \max\{-x, 0\}$.

Hypotheses. $V = V_{\mathscr{A}} + V_e$ where $V_{\mathscr{A}}(x)$ is $C^\infty(\mathbb{R}^n)$, real valued and has an analytic extension to the cone

$$\Gamma_V = \{x \in \mathbb{C}^n : \Re x_1 < -R_V, |\Im x| < \rho_V |\Re x_1|\} \tag{1.1}$$

for some $R_V > 0$ and $\rho_V > 0$, and for some positive constants C and ϵ_V

$$|V_{\mathscr{A}}(x)| \le C\langle \Re x_1\rangle^{-\epsilon_V} \quad \text{for } x \in \Gamma_V \cup \mathbb{R}^n \tag{1.2}$$

$$\lim_{|x|\to\infty} V_{\mathscr{A}}(x) = 0. \tag{1.3}$$

The other term is $V_e = e^{\mu_V x_1^-} V_0$ where $\mu_V > 0$ and V_0 is symmetric and H_0-compact and commutes with any operator which is multiplication by a function of x_1.

Example. Let $V_{\mathscr{A}}(x) = \langle x_1\rangle^{-\epsilon_V}(1+\ln\langle x\rangle)^{-1}$ and suppose that $V_0(x)$ is a real valued function that can be expressed as a sum of a function in $L^p(\mathbb{R}^n)$, $p > \max\{2, n/2\}$ and a bounded function $V_\infty(x)$ with $\lim_{|x|\to\infty} V_\infty(x) = 0$ then $V = V_{\mathscr{A}} + e^{-\mu_V x_1^-} V_0$ satisfies the Hypotheses. (Here V_0 acts by multiplication.) See Perry (1983, Section 19.1) for a more detailed discussion of Stark operators. The Coulomb potential $V(x) = 1/|x|$ is a second example if $n \ge 3$. However, in the present context, if $F > 0$ then the Coulomb potential is "short range" as will be explained below and Theorems 1 and 2 below are not new for short range scattering; see White (2001).

The results here are new in the case of long range potentials V. Here "long range" means that the usual (Møller) wave operators (see (1.8) below) of scattering theory do *not* exist. In the present context that corresponds, roughly to $0 < \epsilon_V \le 1/2$ but a precise criterion is given in §4; see (4.2). Consequently some generalized notion of wave operator is needed. The wave operators most convenient to our purposes are the two Hilbert space wave operators

$$W_J^{\pm} = \text{s-}\lim_{t\to\pm\infty} e^{itH} J e^{-itH_0} \tag{1.4}$$

where J is a bounded operator on $L^2(\mathbb{R}^n)$ to be chosen in the next paragraph and "s-lim" means the limit is taken in the strong topology.

Following Isozaki, Kitada, and Yajima (Isozaki, 1982; Isozaki and Kitada, 1985a,b) and Kitada and Yajima (1982, 1983) (who used the terminology "time independent modifier" for J), we define the Fourier integral operators

$$J^{\pm}u(x) = \int e^{i\phi^{\pm}(x,\xi)} a^{\pm}(x,\xi)\hat{u}(\xi)\, d_1\xi. \tag{1.5}$$

where integrals are over $\mathbb{R}^n$ unless otherwise indicated and d_1x is $(2\pi)^{-n/2}$ times Lebesgue measure on $\mathbb{R}^n$. The phase $\phi^\pm$ and symbol $a^\pm$ will be introduced in §2 and §3 respectively and are very similar to those constructed in White (2001). The operator J is defined so that it agrees with J^+ (resp. J^-) on outgoing (resp. incoming) states (see (2.5) below). It is known (White, 1990) that $W_J^\pm$ exists (that is the limit in (1.4) exists) and is "complete" under more general assumptions than those in the Hypotheses above.

We shall consider the scattering operator $S_J = (W_J^+)^* W_J^-$. The scattering matrix $S_J(\lambda)$ is defined by restricting S_J to manifolds of constant free energy λ. These manifolds are hyperplanes $\mathbb{R}^{n-1}$ in the present context ($F > 0$) whereas, for comparision, they are spheres $\mathbb{S}^{n-1}$ if $F = 0$. The restriction operator, $T_0(\lambda)$ is constructed explicitly after (5.9); it is a mapping from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{n-1})$. Formally we have $\lambda T_0(\lambda) = T_0(\lambda)H_0$ and $S_J(\lambda)T_0(\lambda) = T_0(\lambda)S_J$.

Before considering the extension of the scattering matrix we show that the resolvent operator $R(z) = (H - z)^{-1}$, has an extension from $\mathbb{C}_\pm = \{z \in \mathbb{C} : \pm\Im z > 0\}$ to $\mathbb{C}$ in a certain context. We shall need the notation

$$V_J = HJ - JH_0$$

for the "effective potential;" later we shall see that V_J is a bounded operator. Also V_J^+ (resp. V_J^-) will be a certain restriction of V_J to outgoing (resp. incoming) states and $V_J = V_J^+ + V_J^-$; see (2.2) below.

Theorem 1. *For any $\mu > 0$, each of the 9 operators*

$$e^{-\mu x_1^-} R(z) e^{-\mu x_1^-}, \quad e^{-\mu x_1^-} R(z) V_J^\pm, \quad (V_J^\pm)^* R(z) e^{-\mu x_1^-},$$
$$(V_J^\pm)^* R(z) V_J^\pm \quad \textit{and} \quad (V_J^\mp)^* R(z) V_J^\pm$$

has a meromorphic extension from $\mathbb{C}_+$ (resp. $\mathbb{C}_-$) to $\mathbb{C}$ as a bounded operator on $L^2(\mathbb{R}^n)$.

The main result of this paper is:

Theorem 2. *For almost every real λ*

$$S_J(\lambda) - 1 = -2\pi i T_0(\lambda) J^* V_J T_0(\lambda)^* + 2\pi i T_0(\lambda) V_J^* R(\lambda + i0) V_J T_0(\lambda)^* \tag{1.6}$$

and $S_J(\lambda)$ has a meromorphic extension in λ to all of $\mathbb{C}$ as a bounded operator on $L^2(\mathbb{R}^{n-1})$. Moreover $T_0(\lambda) J^* V_J T_0(\lambda)^*$ has a holomorphic extension to $\mathbb{C}$.

As a consequence of this theorem, a pole of the (extended) scattering matrix $S_J(\lambda)$ must be a pole of the (extended) resolvent $R(z)$. That is to say that, a pole of $S_J(\lambda)$ which is what physicists refer to as a resonance, is a pole of the resolvent operator which is what mathematicians define to be a resonance. The converse is not clear, to the author at least. In the case of Schödinger operators ($F = 0$), the two concepts of resonance coincide (Gérard and Martinez, 1989).

Theorems 1 and 2 are extensions to the long range case of earlier works (Hislop and White, 1999; White, 2001).

Significantly Theorem 2 differs from the comparable result for Schrödinger operators ($F = 0$) of Gérard and Martinez (1989) and Agmon and Klein (1992). In those results, again the scattering matrix does indeed extend (to a cone in $\mathbb{C}$) under roughly comparable hypotheses on the potential V. However the extension is as a bounded operator (on $L^2(\mathbb{S}^{n-1})$ in their case) only if the potential V is short range. If V is long range then the extension is as bounded operators on Gevrey spaces $G^s(\mathbb{S}^{n-1})$ for a restricted range of the parameter s with the restriction depending on the rate of decay of the potential. In Agmon and Klein (1992), very slowly decaying potentials are considered and for them even the Gevrey spaces are not adequate for the extension. No such problem arises here in the case $F > 0$: the extension exists as a bounded operator in $L^2(\mathbb{R}^{n-1})$. Other works on continuation may be found in Bommier (1994), Sigal (1986), and, in the Start case, Yajima (1981).

The notion of scattering matrix employed by Gérard and Martinez (1989) is the same as here: $S_J(\lambda)$ where J is chosen comparably to here. In Agmon and Klein (1992), potentials are assumed to be spherically symmetric and the scattering matrix is defined in terms of generalized eigenfunctions. Certainly some generalized notion of the scattering matrix of short range scattering is needed to treat long range scattering and $S_J(\lambda)$ suffices but it is natural to ask how $S_J(\lambda)$ compares to other, more historical definitions of scattering matrix. Therefore we will compare $S_J(\lambda)$ to the scattering matrix associated with the Dollard modified wave operators which we shall now introduce.

Consider therefore the modified wave operators which were applied by Dollard (1964) to the study of scattering by a Coulomb potential (when $F = 0$). The modified wave operators are defined by

$$\Omega_\gamma^\pm = \text{s-}\lim_{t\to\pm\infty} e^{itH} e^{-itH_0} e^{-i\gamma_t(D)} \tag{1.7}$$

where $e^{-i\gamma_t(D)}$ is equivalent by way of the Fourier transform to multiplication by a function $e^{-i\gamma_t(\xi)}$ ($\xi \in \mathbb{R}^n$) satisfying certain conditions; see §4. Both the modified and two Hilbert space wave operators (1.4) are generalizations of the Møller wave operators:

$$\Omega_0^\pm = W_\mathbf{1}^\pm = \text{ s-}\lim_{t\to\pm\infty} e^{itH} e^{-itH_0} \tag{1.8}$$

Corresponding to the modified wave operators is the scattering operator defined by $\Sigma_\gamma = (\Omega_\gamma^+)^* \Omega_\gamma^-$. The scattering matrix $\Sigma_\gamma(\lambda)$ can be defined by restricting Σ_γ to a manifold of free energy λ in the same manner as $S_J(\lambda)$ was defined by restricting S_J.

How should γ_t be chosen? In terms of the phase ϕ^+ to be defined in §2, we define $\theta^+(x, \xi) = \phi^+(x, \xi) - x \cdot \xi$ and

$$\gamma_t(\xi) = \mp\theta^+(-R_\gamma \mathbf{e}_1 \pm 2t\xi - Ft^2\mathbf{e}_1, \pm\xi - Ft\mathbf{e}_1) \tag{1.9}$$

if $\pm t > 0$ and $R_\gamma > 0$ is the constant of (2.6) below. Then we have

Theorem 3. *The modified wave operators $\Omega_\gamma^\pm$ of* (1.7) *where the modifier is defined by* (1.9) *exist and are complete and moreover*

$$\Omega_\gamma^\pm = W_J^\pm.$$

The Møller wave operators $W_\mathbf{1}^\pm$ (1.8) *exist if $\epsilon_V > 1/2$ or, more precisely, if and only if condition* (4.2) *is valid. If $\epsilon_V > 1/2$ then $W_\mathbf{1}^\pm = W_J^\pm = \Omega_\gamma^\pm$.*

Corollary 4. *The two definitions of the scattering matrix $\Sigma_\gamma(\lambda) = S_J(\lambda)$ coincide. The scattering matrix has a meromorphic extension to the entire complex plane as a bounded operator on $L^2(\mathbb{R}^{n-1})$. If $\epsilon_V > 1/2$ then the scattering matrix $\Sigma_0(\lambda) = S_{\mathbf{1}}(\lambda)$, associated with the Møller wave operators, has such an extension.*

Proof of Corollary 4. This result follows from Theorems 2 and 3 immediately. □

The plan for the subsequent sections is this: In §2, we overview differences between the long range case here, compared to the short range case of White (2001). Additionally, in §2 the definition of the phase $\phi^\pm$ of $J^\pm$ is given. The construction of the symbol $a^\pm$ is outlined in §3. Theorem 3 follows by the method of stationary phase and a proof is given in §4 assuming Theorems 1 and 2. In §5 groundwork is laid for the proof of Theorems 1 and 2 and the fundamental analytic extension result 2 is stated. Theorem 2 is derived from Theorem 1 in §6; Theorem 1 is derived in §7 with the help of Lemma 7.2. Finally in §8, Lemma 7.2 is derived showing the use of analytic properties of $\phi^\pm$ and $a^\pm$. To avoid lengthy repetitions of arguments the reader will be referred to White (2001) when no confusion should arise but most results are stated completely here.

2. Phase

In this section we give the strategy for the proofs of Theorems 1 and 2 of §1 and highlight the differences between the short range case considered in White (2001) and the long range case considered here. Then the choice of the phase function of the operators $J^\pm$ is described in some detail.

The long and short range cases are not so different as might be expected because even in the short range case (White, 2001), long range methods (such as using the wave operators $W_J^\pm$ of (1.4)) were used. The same is true of the Schrödinger case ($F = 0$): Gérard and Martinez (1989) also use $W_J^\pm$ to treat both long and short range potentials. Therefore the proof of Theorems 1 and 2 of §1 can parallel that of the comparable results in White (2001). We shall introduce the "effective" potential in an effort to explain the differences between the short and long range cases. First we will need operators which act as projections onto the incoming and outgoing subspaces. However true projection operators have "bad commutator properties" and so we proceed as follows. We define $\tilde{\chi} \in C^\infty(\mathbb{R})$ to be a smooth version of the Heaviside function:

$$\tilde{\chi}(\xi_1) = \begin{cases} 1 & \text{if } x_1 > 1/2 \\ 0 & \text{if } x_1 < -1/2 \end{cases} \tag{2.1}$$

and so that $\tilde{\chi}(\xi_1)^2 + \tilde{\chi}(-\xi_1)^2 = 1$ and $\tilde{\chi}(\xi_1) \geq 0$. Our "approximate projections" onto the incoming (resp. outgoing) state space are defined, for any $\kappa > 0$, by $\tilde{\chi}(D_1/\kappa)$(resp. $\tilde{\chi}(-D_1/\kappa)$.) where $D_1 = -i\partial/\partial x_1$. Therefore $\tilde{\chi}(D_1/\kappa)$ is equivalent via the Fourier transform to multiplication by $\tilde{\chi}(\xi_1/\kappa)$. The effective potential is $V_J = \sum_\pm V_J^\pm \equiv V_J^+ + V_J^-$ where $V_J^\pm = HJ^\pm\tilde{\chi}(\mp D_1/\kappa) - J^\pm\tilde{\chi}(\mp D_1/\kappa)H_0$ or

$$V_J^\pm = [H_{\mathscr{A}}J^\pm - J^\pm H_0]\tilde{\chi}(\mp D_1/\kappa) + V_e J^\pm\tilde{\chi}(\mp D_1/\kappa) \mp \frac{iF}{\kappa}J^\pm\tilde{\chi}'(\mp D_1/\kappa) \tag{2.2}$$

where $H_{\mathscr{A}} = H_0 + V_{\mathscr{A}}$ and $\tilde{\chi}'$ is the derivative of $\tilde{\chi}$. By a formal calculation that can easily be justified $H_{\mathscr{A}} J^{\pm} - J^{\pm} H_0$ is a Fourier integral operator like $J^{\pm}$ and with the same phase $\phi^{\pm}$ but with symbol

$$\tilde{t}^{\pm}(x, \xi) = -2i\nabla_x \phi^{\pm}(x, \xi) \cdot \nabla_x a^{\pm}(x, \xi) + iF \frac{\partial a^{\pm}}{\partial \xi_1}(x, \xi) \\ + p^{\pm}(x, \xi) a^{\pm}(x, \xi) - \Delta_x a^{\pm}(x, \xi) \tag{2.3}$$

where

$$p^{\pm}(x, \xi) = |\nabla_x \phi^{\pm}(x, \xi)|^2 - F \frac{\partial \phi^{\pm}}{\partial \xi_1}(x, \xi) + F x_1 + V_{\mathscr{A}}(x) - |\xi|^2 - i\Delta_x \phi^{\pm}(x, \xi). \tag{2.4}$$

We shall want $V_J^{\pm}$ to be small in some sense and we choose $\phi^{\pm}$ in §2 and $a^{\pm}$ in §3 so that $\tilde{t}^{\pm}(x, \xi) = O(e^{-\mu x_1^-})$, for some $\mu > 0$, provided $\mp \xi_1 \gg 0$. A similar choice was made in the short range case. However, the effective potential also includes terms like the last term on the right hand side of (2.2) which arises as the commutator of H_0 with the incoming or outgoing approximate projections. The symbols of these terms are compactly supported in the ξ_1 variable. It will also be convenient if their operator norms are small as well. Consequently we shall eliminate any cutoff of the phase and choose the parameter $\kappa > 0$ in (2.2) to be large. Large κ corresponds to a gradual transition from incoming to outgoing and *vice versa*. Consequently our estimates of the phase ϕ^+ (resp. ϕ^-) must be uniform even into the incoming region $\xi_1 > 0$ (resp. outgoing region $\xi_1 < 0$). Furthermore, since the phase $\phi^{\pm}$ is not cutoff, we are forced to work with J^+ and J^- separately instead of the operator J of White (2001). The appropriate definition of J here for the statement of the results in §1 is

$$J = J^+ \tilde{\chi}(-D_1/\kappa) + J^- \tilde{\chi}(D_1/\kappa). \tag{2.5}$$

This completes the overview of the difficulties specific to the long range case. The results in White (2001) that used the short range assumption $\epsilon_V > 1/2$ in a significant way are Lemma 5.1 and Theorem 6.2, there; these correspond to Lemma 5.1 below and Theorem 1 above, respectively.

Next we introduce the phase function $\phi^{\pm}(x, \xi)$ of $J^{\pm}$. We set $\phi^-(x, \xi) = -\phi^+(x, -\xi)$ and therefore need only define the outgoing portion ϕ^+. The construction here is the same as in White (2001) but omits any cutoff of the incoming portion ($\xi_1 \gg 0$); it is preferable to cutoff only the symbol and not the phase.

We define now ϕ^+, or more conveniently $\theta^+ = \phi^+(x, \xi) - x \cdot \xi$. The motivation here is that p^+ of (2.4) should decay rapidly in the field direction $-\mathbf{e}_1$. As a first approximation of θ^+ we set

$$\theta_1(x, \xi) = \begin{cases} \displaystyle\int_0^\infty V_{\mathscr{A}}(x + 2t\xi - Ft^2\mathbf{e}_1) - V_{\mathscr{A}}(-R_\gamma \mathbf{e}_1 + 2t\xi_\perp - Ft^2\mathbf{e}_1)dt & \text{if } \epsilon_V \le 1/2 \\ \displaystyle\int_0^\infty V_{\mathscr{A}}(x + 2t\xi - Ft^2\mathbf{e}_1)dt & \text{if } \epsilon_V > 1/2 \end{cases} \tag{2.6}$$

for any fixed $R_\gamma > R_V$. (This is the same R_γ that appears in (1.9).) This definition corrects the definition of θ_1 given in White (2001) in the short range case $\epsilon_V > 1/2$.

The integral defining θ_1 exists, as an improper Riemann integral, if $\epsilon_V \leq 1/2$ and absolutely if $\epsilon_V > 1/2$ and defines a differentiable function of (x, ξ) whose derivative can be calculated by differentiating under the integral sign. Introduce a cutoff function $\tilde{\chi}_{(-\infty,1)} \in C^\infty(\mathbb{R})$ function so that

$$\tilde{\chi}_{(-\infty,1)}(x_1) = \begin{cases} 1 & \text{if } x_1 < 1/2 \\ 0 & \text{if } x_1 > 1. \end{cases}$$

We further define

$$\psi(x, \xi) = \tilde{\chi}_{(-\infty,1)}(\Re x_1)|\nabla_x \theta_1(x, \xi)|^2$$
$$\theta_2(x, \xi) = \theta_1(x, \xi) + \int_0^\infty \psi(x + 2t\xi - Ft^2\mathbf{e}_1, \xi - Ft\mathbf{e}_1)dt$$

This is the same definition as used in White (2001) if $\Re\xi_1 < 0$ (that is, in the important outgoing region) but here we omit any cutoff in ξ_1 in the definition of ψ. Define θ^+ in terms of θ_2 as follows: let $\tilde{\chi}_{(-\infty,-R_0)}$ be a $C^\infty(\mathbb{R})$ function which is $\tilde{\chi}_{(-\infty,-R_0)}(x_1) = 0$ if $x_1 > -R_0$ and $\tilde{\chi}_{(-\infty,-R_0)}(x_1) = 1$ if $x_1 < -R_0 - 1$. Let

$$\omega_0(x_1) = (x_1 + R_0 + 1)\tilde{\chi}_{(-\infty,-R_0)}(\Re x_1) - R_0 - 1$$

and

$$\theta^+(x_1, x_\perp, \xi) = \theta_2(\omega_0(x_1), x_\perp, \xi). \tag{2.7}$$

This cutoff locks in the analyticity of θ^+ in the other variables when $x_1 > -R_0$ is fixed. The parameter R_0 will be specified very large in §5 below. This completes the construction of θ^+ and therefore of $\phi^+(x, \xi) = x \cdot \xi + \theta^+(x, \xi)$; their properties are summarized in the proposition below.

To state that proposition, we first recall that $V_{\mathscr{A}}$ is analytic on a cone. Correspondingly the phase ϕ^+ is analytic on

$$\Gamma(\gamma, R, K) = \Big\{(x, \xi) \in \mathbb{C}^{2n} : |\Im x| < -\gamma\rho_V(\Re x_1 + K), \Re x_1 < -R,$$
$$|\Im\xi| < \max\{-\gamma^2\rho_V\Re\xi_1, 1\}, \Re\xi_1 < \sqrt{(F/2)\Re x_1^-}\Big\} \tag{2.8}$$

where $0 < \gamma \leq 1$, and provided $R > K > 0$ are adequately large. Intuitively $1 - \gamma$, $1/R$, $1/K$ are small positive constants to be chosen as appropriate for the proof of the proposition below and for the construction of the symbol in §3 below. We introduce the notation $C_b^m(\Omega)$, $0 \leq m \leq \infty$ for those functions in $C^m(\Omega)$ which are bounded along with all their partial derivatives up to order m on a set Ω in $\mathbb{R}^n$ or $\mathbb{C}^n$.

Proposition 1. *Let $0 < \gamma_\phi < 1$ be a given constant. Then there is R_ϕ and K_ϕ, $0 < K_\phi < R_\phi$ so that the phase, $\phi^+(x, \xi)$, as defined above, with $R_0 > R_\phi$ in (2.7) adequately large exists and is $C^\infty(\Omega)$ where $\Omega = \Gamma \cup \{(x, \xi) \in \mathbb{R}^{2n} : \xi_1 < \sqrt{(F/2)x_1^-}\}$ and $\Gamma \equiv \Gamma(\gamma_\phi, R_\phi, K_\phi)$. Moreover $\phi^+(x, \xi)$ is real valued when x and ξ are; ϕ^+ is holomorphic on $\Omega \cap \{(x, \xi) : \Re x_1 < -R_0 - 1\}$. If $-R_0 - 1 \leq \Re x_1 \leq -R_0$ (resp. $\Re x_1 >$*

$-R_0$) is fixed then $\phi^+(x,\xi)$ is holomorphic in the variables $(x_\perp,\xi)$ on the cross section $\Gamma(x_1)=\{(x_\perp,\xi):(x_1,x_\perp,\xi)\in\Gamma\}$ (resp. on $\Gamma(-R_0)$) of Γ. Further, for any $\alpha,\beta\in\mathbb{N}_0$

$$\sup\{|D_x^\alpha D_\xi^\beta[\nabla_x\phi^+(x,\xi)-\xi]|:(x,\xi)\in\Omega\}=o(1)\quad\text{as } R_0\to 0. \tag{2.9}$$

Moreover for any ϵ_ϕ, $0<\epsilon_\phi<\epsilon_V$, there is $C>0$ so that

$$|\nabla_x\phi^+(x,\xi)-\xi|<C|\Re x_1|^{-\epsilon_\phi}(|\Re x_1|+(\Re\xi_1)^2)^{-1/2}; \tag{2.10}$$

$$|p(x,\xi)|<C|\Re x_1|^{-1-\epsilon_\phi}(|\Re x_1|+(\Re\xi_1)^2)^{-1/2} \tag{2.11}$$

for all $(x,\xi)\in\Gamma$. The functions p and $\nabla_x\phi^+-\xi$, and $\nabla_\xi\phi^+-x$, belong to $C_b^\infty(\Omega)$ and for any real R and all $\alpha,\beta\in\mathbb{N}_0^n$

$$\lim_{|(x,\xi)|\to\infty}\chi_{(-\infty,R)}(x_1)[|D_x^\alpha D_\xi^\beta[\nabla_x\phi^+(x,\xi)-\xi]|+|D_x^\alpha D_\xi^\beta p(x,\xi)|]=0,$$

the limit being taken inside Ω. Finally, if $\epsilon_V>1/2$ in the Hypothesis, then $\phi^+(x,\xi)-x\cdot\xi$ is in $C_b^\infty(\Omega)$ and $\phi^+(x,\xi)-x\cdot\xi=O(|\Re x_1|^{-\epsilon_V+1/2})$ on Γ.

Remark. This is (White, 2001, Proposition 2.1) except that Ω has been enlarged and the additional conclusion $\nabla_\xi\phi^+(x,\xi)-x$ is in $C_b^\infty(\Omega)$ has been drawn. Later in (5.8) we shall need estimates for the first few derivatives of $\phi^+(x,\xi)$ on the outgoing region $\Re\xi_1<\kappa$ where $\kappa>0$ is large. It will be possible to fix κ as large as necessary provided $-\Re x_1$ is restricted to be suitably large because the estimates above are uniform on Ω or Γ.

Remarks on the Proof of the Proposition. We begin with the observation that the derivatives of $V_{\mathscr{A}}$ satisfy, for each multi-index α in $\mathbb{N}_0^n$

$$D^\alpha V_{\mathscr{A}}(x)=O(\langle\Re x_1\rangle^{-\epsilon_V-|\alpha|})\quad\text{for } x\in\Gamma_V. \tag{2.12}$$

When $\alpha=0$ this is just the Hypotheses; the general case follows from the Cauchy integral formula for derivatives of analytic functions and the conical shape of Γ_V. Similarly the estimate (2.10) implies the estimate (2.9) on Γ.

The proof is very similar to that of White (2001, Proposition 2.1) and the reader is referred there. The enlargement of Ω and Γ means that $\Re\xi_1$ may be positive or more precisely $0<\Re\xi_1<\sqrt{F\Re x_1^-/2}$ so that $\Re x_1$ must be compensatingly negative. In this region the characteristic curves $x+2t\xi-Ft^2\mathbf{e}_1$ have first component $x_1+\xi_1^2/F-F(t-\xi_1/F)^2$ and the real part satisfies $\Re x_1+\Re\xi_1^2/F<\Re x_1/2<0$. Therefore whenever $-\Re x_1>0$, is large enough then the characteristic curve lies entirely within Γ_V and they estimates (2.10) and (2.11) can be verified as in White (2001, Proposition 2.1).

The boundedness of $\nabla_\xi\theta^+(x,\xi)$ and its derivatives follows by way of White (2001, Lemma 2.2). The remaining conclusions differ from those in White (2001, Proposition 2.1) only because Ω has been enlarged and the verification is the same. $\square$

3. Symbol

In this section we complete the specification of the operator $J^{\pm}$ (1.5) by indicating the choice of symbol $a^{\pm}$, the phase $\phi^{\pm}$ being already specified in §2. Unlike the phase, the symbol is constructed exactly as in White (2001) and so it is only necessary to recall the relevant results for reference. One critical criterion in the choice of the symbol a^{+} of $J^{\pm}$ is that the symbol $\tilde{t}^{+}$ of equation (2.3) of $H_{\mathscr{A}}J^{+} - J^{+}H_0$ will decay to 0 exponentially fast as $\Re x_1 \to -\infty$ in the outgoing regions of phase space. We shall find it convenient to construct the restriction b of a^{+} to an outgoing region of space and to obtain a^{+} by simply extending b by (3.8) below. Therefore define t^{+} by (2.3) with $\tilde{t}^{+}$ and a^{+} there replaced by t^{+} and b respectively. Regrettably $b(x, \xi)$, as constructed below, is not analytic in x_1 and so, in order to keep track of the asymptotic behavior of b and its derivatives, we introduce the notation

$$\frac{\partial}{\partial x_j} = \frac{1}{2}\left(\frac{\partial}{\partial \Re x_j} - i\frac{\partial}{\partial \Im x_j}\right)\frac{\partial}{\partial \bar{x}_j} = \frac{1}{2}\left(\frac{\partial}{\partial \Re x_j} + i\frac{\partial}{\partial \Im x_j}\right) \tag{3.1}$$

for $1 \le j \le n$. For $\alpha \in \mathbb{N}_0^n$ define $\partial_x^{\alpha} = \partial^{\alpha_1}/\partial x_1^{\alpha_1} \ldots \partial^{\alpha_n}/\partial x_n^{\alpha_n}$; $\bar{\partial}_x^{\alpha}$ is defined similarly. The symbol b has the following properties.

Proposition 2. *For any γ_σ, $0 < \gamma_\sigma < \gamma_\phi$, and $\kappa_\sigma > 0$ there exist constants $R_\sigma > K_\sigma > 0$, and a function $b(x, \xi)$ defined and C^∞ on $\Gamma \equiv \Gamma(\gamma_\sigma, R_\sigma, K_\sigma) \cap \{(x, \xi) \in \mathbb{C}^{2n} : \Re\xi_1 < -\kappa_\sigma\}$ such that, for fixed x_1, $\Re x_1 < -R_\sigma$, $b(x, \xi)$ is holomorphic in the remaining $2n - 1$ variables on (the cross section of) Γ. Also, for all $\alpha, \beta \in \mathbb{N}_0^n$, there is $\mu > 0$, and $C_{\alpha,\beta} > 0$ so that,*

$$\left|\bar{\partial}_x^{\alpha}\partial_x^{\beta} t^{+}(x, \xi)\right| \le C_{\alpha,\beta} e^{-\mu\langle \Re x_1\rangle}; \tag{3.2}$$

$$\lim_{|(x,\xi)| \to \infty} \bar{\partial}_x^{\alpha}\partial_x^{\beta} t^{+}(x, \xi) = 0 \tag{3.3}$$

$$|\Re x_1|^{\alpha_1+\beta_1+1/2+\epsilon_\phi}(|\Re x_1| + (\Re\xi_1)^2)^{1/2}|\bar{\partial}_x^{\alpha}\partial_x^{\beta}(b(x, \xi) - 1)| \le C_{\alpha,\beta}; \tag{3.4}$$

$$\lim_{|(x,\xi)| \to \infty} |\Re x_1|^{\alpha_1+\beta_1+1/2}(|\Re x_1| + (\Re\xi_1)^2)^{1/2}\left|\bar{\partial}_x^{\alpha}\partial_x^{\beta}(b(x, \xi) - 1)\right| = 0. \tag{3.5}$$

Moreover, there is $\delta > 0$ ($\delta = \delta(\mu, \alpha, \beta)$) so that on $\Gamma_\delta \equiv \Gamma \cap \{(x, \xi) \in \mathbb{C}^{2n} : |\Im x_\perp|^2 + |\Im\xi|^2 < \delta^2, \Re\xi_1 < 3\kappa_\sigma/2\}$ b can written as $b = b_{\mathscr{A}} + b_e$ where $b_{\mathscr{A}}$ is analytic in all $2n$ variables on Γ_δ and

$$\bar{\partial}_x^{\alpha}\partial_x^{\beta} b_e(x, \xi) < C_{\alpha,\beta} e^{-\mu\langle \Re x_1\rangle}$$

Here γ_ϕ and ϵ_ϕ are constants of Proposition 1; $0 < \gamma_\phi < 1$, $0 < \epsilon_\phi < \epsilon_V$.

Proposition 1 is Proposition 3.1 and Corollary 4.2 of White (2001) combined and a proof can be found there. The symbol $a^{\pm}$ of $J^{\pm}$ can be defined in terms of b as follows. First extend b to the region $\Re x_1 > R_\sigma$ by replacing b by

$$1 + \tilde{\chi}_{(-\infty,-2)}(\Re x_1/R)[b(x, \xi) - 1] \tag{3.6}$$

where

$$\tilde{\chi}_{(-\infty,-2)}(x_1) = \begin{cases} 1 & \text{if } x_1 < -5/2 \\ 0 & \text{if } x_1 > -3/2 \end{cases} \tag{3.7}$$

and where $R > R_\sigma$ is a parameter to be chosen later. The extended function will also be called b. Let

$$a^+(x,\xi) = 1 + \tilde{\chi}_{(-\infty,-2)}(\Re\xi_1/\kappa_\sigma)(b(x,\xi) - 1) \tag{3.8}$$

and similarly define $a^+_{\mathscr{A}}$: replace a^+ in (3.8) by $a^+_{\mathscr{A}}$ and b by $b_{\mathscr{A}}$. Define a^+_e so that $a^+ = a^+_{\mathscr{A}} + a^+_e$. Define the incoming analogues of a^+ and ϕ^+ by "time reversal":

$$a^-(x,\xi) = \overline{a^+(\bar{x}, -\bar{\xi})} \quad \phi^-(x,\xi) = -\phi^+(x,-\xi). \tag{3.9}$$

(Recall ϕ^+ is real valued on $\mathbb{R}^{2n}$.) Define $a^-_{\mathscr{A}}$ (resp. a^-_e) similarly replace $a^\pm$ in (3.9) by $a^\pm_{\mathscr{A}}$ (resp. $a^\pm_e$). These definitions agree with those in White (2001); however the introduction of a combined phase ϕ in White (2001) was misguided.

4. Modified Wave Operators

In this section we shall derive Theorem 3 from Theorem 2. We begin with some introductory comments about the modified wave operators Ω_γ of (1.7). The choice of γ_t given in (1.9) is made to facilitate the proof of Theorem 3. The one restriction is that the resultant wave operators $\Omega^\pm_\gamma$ should "intertwine" H and H_0 (Hörmander, 1976). Since that is an immediate consequence of Theorem 3 and the intertwining principle for $W^\pm_J$ (see Yafaev, 1992) we shall not stop to find a direct proof.

We also recall that γ_t in the definition of the modified wave operators is not unique. To be more precise we state the following result of Hörmander (1976, Theorem 3.1). *Suppose that the modified wave operators* $\Omega^\pm_\gamma$ and $\Omega^\pm_{\tilde{\gamma}}$ both exist with two different choices of γ_t and $\tilde{\gamma}_t$. Then the range of $\Omega^\pm_{\tilde{\gamma}}$ is contained in the range of $\Omega^\pm_\gamma$ if and only if $\exp(-i(\gamma_t - \tilde{\gamma}_t))$ converges in measure as $t \to \pm\infty$ to some function, Φ say and, in this event $\Omega^\pm_\gamma = \Omega^\pm_{\tilde{\gamma}}\Phi(D)$.

The relationship between the modified wave operators $\Omega^\pm_\gamma$ and the two Hilbert space wave operators $W^\pm_J$ was studied by Kitada (1987) in the Schrödinger case ($F = 0$). There, however the choice of γ_t is the historical one and is related to the solution of a certain Hamilton–Jacobi equation; here γ_t can be chosen as is convenient for the proof of Theorem 3.

Proof of Theorem 3. Let us begin by assuming the first statement and showing how it implies the final statement about the Møller wave operators $W^\pm_{\mathbf{1}} = \Omega^\pm_0$. We apply Hörmander (1976, Theorem 3.1), as stated in the preceding paragraph with $\tilde{\gamma}_t \equiv 0$. Since $W^\pm_J$ is complete, that is the range is equal to the subspace of continuity of H (White, 1990), and since, $\Omega^\pm_0$ has range inside the subspace of continuity of H by a standard argument (Kato, 1980), that result applies and we conclude: $\Omega^\pm_0$, exists if and only if $\exp(i\gamma_t)$ converges in measure. Recall therefore the definition of γ_t, $t > 0$

(1.9), first in the case that $0 < \epsilon_V \leq 1/2$

$$
\begin{aligned}
\gamma_t(\xi) &= -\theta^+(-R\mathbf{e}_1 + 2t\xi - Ft^2\mathbf{e}_1, \xi - Ft\mathbf{e}_1) \\
&= \int_t^\infty V_{\mathscr{A}}(-R\mathbf{e}_1 + 2\tau\xi - F\tau^2\mathbf{e}_1) - V_{\mathscr{A}}(-R\mathbf{e}_1 + 2\tau\xi_\perp - F\tau^2\mathbf{e}_1) \\
&\quad + |\nabla_x\theta_1(-R\mathbf{e}_1 + 2\tau\xi - F\tau^2\mathbf{e}_1, \xi - F\tau\mathbf{e}_1)|^2\, d\tau \\
&\quad - \int_0^t V_{\mathscr{A}}(-R\mathbf{e}_1 + 2\tau\xi_\perp - F\tau^2\mathbf{e}_1)d\tau \qquad (4.1)
\end{aligned}
$$

provided that $-R - 2t\xi_1 - Ft^2$ is sufficiently large and negative so that we are in the region where $\theta^+ = \theta_2$. In this way we see that the Møller wave operator $W_\mathbf{1}^\pm$ exists if and only if

$$\exp\left(i\int_0^t V_{\mathscr{A}}(2\tau\xi_\perp - F\tau^2\mathbf{e}_1)d\tau\right) \text{ converges in measure as } t \to \pm\infty. \qquad (4.2)$$

(We omit the term $-R\mathbf{e}_1$ from the argument to simplify the statement.) A similar calculation applies when $\epsilon_V > 1/2$, but in this case $\gamma_t \to 0$ as $t \to \pm\infty$ which implies that the modified wave operators and Møller wave operators coincide: $\Omega_\gamma^\pm = W_\mathbf{1}^\pm$.

It remains to prove the first statement of Theorem 3 and that can be done by a stationary phase argument which we now outline. Suppose that $u \in L^2(\mathbb{R}^n)$ has Fourier transform $\hat{u} \in C_0^\infty(\{\xi : |\xi - \xi_0| < \eta\})$ where $\xi_0 \in \mathbb{R}^n$ is arbitrary and $\eta > 0$ is suitably small (as specified below). Such u form a fundamental subset of $L^2(\mathbb{R}^n)$ (that is, their linear span is dense). It will be shown that

$$Je^{-itH_0}u - e^{-itH_0 - i\gamma_t}u \to 0 \quad \text{as } t \to \infty \qquad (4.3)$$

with convergence in $L^2(\mathbb{R}^n)$. This will show that Ω_γ^+ exists and is W_J^+, which is known to exist (White, 1990) and this will conclude the proof.

Recall the Avron–Herbst formula (Avron and Herbst, 1977; Perry, 1983)

$$e^{-itH_0} = e^{-iF^2t^3/3}e^{-itFx_1}e^{iD_1Ft^2 - it(-\Delta)}$$

so that

$$Je^{-itH_0}u(x) = e^{-iF^2t^3/3}e^{-itFx_1}\int e^{ix\cdot\xi + i\theta(x, \xi - Ft\mathbf{e}_1)}a(x, \xi - Ft\mathbf{e}_1)e^{i\xi_1Ft^2 - it|\xi|^2}\hat{u}(\xi)d_1\xi$$

and similarly

$$e^{-itH_0 - i\gamma_t(D)}u(x) = e^{-iF^2t^3/3}e^{-itFx_1}\int e^{ix\cdot\xi + i\xi_1Ft^2 - it|\xi|^2 - i\gamma_t(\xi)}\hat{u}(\xi)\, d_1\xi.$$

To verify 4.3, it suffices to prove that $v_J(\cdot, t) - v_\gamma(\cdot, t)$ converges to 0 in $L^2(\mathbb{R}^n, dx)$ as $t \to \infty$ where

$$
\begin{aligned}
v_J(x, t) &= \int e^{ix\cdot\xi + i\theta(x - Ft^2\mathbf{e}_1, \xi - Ft\mathbf{e}_1)}e^{-it|\xi|^2}a(x - Ft^2\mathbf{e}_1, \xi - Ft\mathbf{e}_1)\hat{u}(\xi)\, d_1\xi \\
v_\gamma(x, t) &= \int e^{ix\cdot\xi - it|\xi|^2 - i\gamma_t(\xi)}\hat{u}(\xi)\, d_1\xi.
\end{aligned}
$$

We apply stationary phase (**?**, §7.7) to v_J and v_γ.

The stationary phase argument is standard and can be left to the reader but the properties of the phase that are needed should be stated. Let us introduce the notation: for each multi-index β $|\beta| \geq 1$

$$D^{\beta}\gamma_{\infty}(\xi_{\perp}) = \int_0^{\infty} (2\tau)^{|\beta|} D^{\beta} V(-R\mathbf{e}_1 + 2\tau\xi_{\perp} - F\tau^2)\, d\tau$$

(The integral may not exist if $\beta = 0$ and $\epsilon_V \leq 1/2$.) Since $|\beta| \geq 1$ the integral exists and can be differentiated by differentiating "under the integral sign" (so that the notation is consistent) and the result is a bounded function of $\xi_{\perp}$. The required estimates are: Given $K > 0$ there is $C > 0$ so that for all $y, \xi \in \mathbb{R}^n$ such that $|y_1| + |\xi| < K$

$$|D^{\beta}\gamma_t(\xi) + D^{\beta}\gamma_{\infty}(\xi_{\perp})| + |D^{\beta}_{\xi}\theta^{+}(ty - Ft^2\mathbf{e}_1, \xi - Ft^2\mathbf{e}_1) + D^{\beta}_{\xi}\gamma_{\infty}(\xi_{\perp})| < Ct^{-\epsilon_V}$$

The differentiation is in the $\xi_{\perp}$ variables but in fact we can allow also differentiation in ξ_1 as well if it is understood that $\partial\gamma_{\infty}/\partial\xi_1 = 0$. The estimates follow directly from the (2.12); Proposition 2.1 is not needed here.

Stationary phase applies to each of the integrals, the one defining v_J and the other v_{γ}. The point of stationary phase for the integral defining v_J is ξ defined implicitly by the equation

$$y - 2\xi + \frac{1}{t}\nabla_{\xi}\theta(ty - Ft^2\mathbf{e}_1, \xi - Ft\mathbf{e}_1) = 0;$$

denote it by $\xi_J(y, t)$. The point of stationary phase for v_{γ} is $\xi_{\gamma}(y, t)$ defined by

$$y - 2\xi - \frac{1}{t}\nabla\gamma_t(\xi) = 0.$$

It is not difficult to show that $\xi_J - \xi_{\gamma} = O(t^{-1-\epsilon_V})$ locally uniformly in y. Comparing the stationary phase expansions of v_J and v_{γ} in t, it follows that the $L^2(\mathbb{R}^n, dx)$ norm of $v_J - v_{\gamma}$ converges to 0 as $t \to \infty$. □

5. Analytic Extensions

In this section we establish a fundamental analytic extension result, Proposition 5.2 in preparation for the proof of Theorems 1 and 2. The proof involves computations with the operators $J^{\pm}$ and $V_J^{\pm}$ or more generally

$$Q^{\pm}u(x) = \int e^{i\phi^{\pm}(x,\xi)} q^{\pm}(x, \xi)\hat{u}(\xi)d\xi \tag{5.1}$$

where the integration is over all of $\mathbb{R}^n$ and u is in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ of smooth functions of rapid decrease. Here the phase $\phi^{\pm}$ is fixed as in §2 but the allowed symbols $q^{\pm}$ will comprise a wide class in $C^{\infty}(\mathbb{R}^{2n})$. The computations also will involve pseudo-differential operators: we introduce therefore

$$\Psi u(x) = \iint e^{i(x-x')\cdot\xi}\psi(x, \xi, x')u(x')dx'\, d\xi \tag{5.2}$$

with $u \in \mathcal{S}(\mathbb{R}^n)$ and the symbol $\psi \in C^\infty(\mathbb{R}^{3n})$. We shall also use the notation $\Psi = \psi(X, D_x, X')$ occasionally. It will be convenient to have a criterion that assures that Ψ is bounded operator on $L^2(\mathbb{R}^n)$ and that is given by the Calderón–Vaillancourt theorem (Calderón and Vaillancourt, 1972), a special case of which will now be stated. Define for each $m = 0, 1, 2, \ldots$

$$|\psi|_m \equiv \sup\{|D_x^\alpha D_{x'}^\gamma D_\xi^\beta \psi(x, \xi, x')|, |\alpha + \beta + \gamma| \leq m, (x, \xi, x') \in \mathbf{R}^{3n}\} \tag{5.3}$$

The version of the Calderón–Vaillancourt theorem (Calderón and Vaillancourt, 1972) (or see Kumano-go, 1981, p. 224, for example) convenient for the present applications states that there is a constant $C > 0$ and integer $m_0 \in \mathbb{N}_0$ so that, for every $\psi \in C^\infty(\mathbb{R}^{3n})$,

$$\|\Psi\| \leq C|\psi|_{m_0}$$

where $\|\cdot\|$ denotes the operator norm on $L^2(\mathbb{R}^n)$.

It is convenient to introduce a class of symbols ψ for the operators to be encountered below. For each $\delta > 0$, let

$$\Omega_\delta = \{(x, \xi, x') \in \mathbb{C}^{3n} : |\Im\xi_1| < \delta, |\Im x_\perp|^2 + |\Im x'_\perp|^2 + |\Im\xi_\perp|^2 < \delta^2,$$
$$|\Im x_1| < \delta \max\{-\Re x_1 - K_\sigma, 1\}, |\Im x'_1| < \delta \max\{-\Re x'_1 - K_\sigma, 1\}\}$$

where $K_\sigma > 0$ is a constant of Proposition 1. For m in $\mathbb{N}_0$, let $C_b^m(\Omega_\delta)$ denote the space of functions $\psi(x, \xi, x')$ which are continuous and bounded along with the partial derivatives up to order m in the $6n$ real variables $\Re x$, $\Im x$ etc. For each $m \in \mathbb{N}_0$ and $\delta, \kappa > 0$, define $\mathcal{B}(m, \delta, \kappa)$ to be the set of all functions ψ in $C_b^m(\Omega_\delta) \cap C^\infty(\Omega_\delta)$ such that, for each fixed x_1 and x'_1, $\psi(x_1, x_\perp, \xi, x'_1, x'_\perp)$ is analytic in the remaining $3n - 2$ complex variables on the cross section of $\Omega_\delta \cap \{|\Re\xi_1| > \kappa\}$.

For any $R > 0$ define $\mathcal{B}_{\mathcal{A}}(m, R, \delta, \kappa) \subset \mathcal{B}(m, \delta, \kappa)$ to include those ψ which are analytic in all $3n$ variables on $\Omega_\delta \cap \{\Re x_1 < -R\} \cap \{\Re x'_1 < -R\} \cap \{|\Re\xi_1| > \kappa\}$ and, for fixed ξ_1, $|\Re\xi_1| \leq \kappa$, $|\Im\xi_1| < \delta$ are analytic in the $3n - 1$ variables $(x, \xi_\perp, x')$ on the cross section of Ω_δ. Define $\mathcal{B} = \bigcup \mathcal{B}(m, \delta, \kappa)$ where the union is over all $m \in \mathbb{N}_0$ and $\delta, \kappa > 0$; define $\mathcal{B}_{\mathcal{A}} = \bigcup \mathcal{B}_{\mathcal{A}}(m, R, \delta, \kappa)$ where this time the union is over all $m \in \mathbb{N}_0$ and $R, \delta, \kappa > 0$. It will also be convenient to say that a function $q^\pm(x, \xi)$ belongs to $\mathcal{B}$ (or $\mathcal{B}_{\mathcal{A}}$) if the function $(x, \xi, x') \mapsto q^\pm(x, \xi)$ does.

Examples. We have $b_{\mathcal{A}}$ and $a_{\mathcal{A}}^\pm$ belong to $\mathcal{B}_{\mathcal{A}}$. To give examples of elements of $\mathcal{B}$ we shall choose, for any $X \geq 0$, $\mu > 0$, h_μ and $h_{\mu,X}$ in $C^\infty(\mathbb{R})$ such that

$$h_\mu(x) = e^{-\mu x^-} \quad \text{if } |x| > 1, \quad \text{and} \quad h_{\mu,X}(x) = h_\mu(x + X) \tag{5.4}$$

so that $h_\mu(x)$ is like $e^{-\mu x^-}$ but smooth at the origin. Then, for any $m \in \mathbb{N}_0$ and $X \geq 0$ there is $\delta, \mu, \kappa > 0$ so that $b_e/h_{\mu,X}$, and $a_e^\pm/h_{\mu,X}$ are in $\mathcal{B}(m, \delta, \kappa)$.

Define further, for any integer $0 \leq m$,

$$\|\psi\|_m = \sup\{|\bar{\partial}^\alpha \partial^\beta \psi(x, \xi, x')| : (x, \xi, x') \in \Omega_\delta, \alpha, \beta \in \mathbb{N}_0^{3n}, |\alpha + \beta| \leq m\}. \tag{5.5}$$

(Reference to δ is omitted from the notation $\|\cdot\|_m$.) Then, for any $\psi \in \mathscr{B}$, we have $\|\psi(X, D_x, X')\| \le C\|\psi\|_{m_0}$, provided m_0 is large enough, by the Calderón–Vaillancourt theorem as stated above,

The operators $J^{\pm}$ may not themselves be bounded. In fact the phase ϕ^+ (resp. ϕ^-) is not well controlled in the incoming region $\Re\xi_1 \gg 0$ (resp. outgoing region $\Re\xi_1 \ll 0$) but the operator $J^+\tilde{\chi}(-D_1/\kappa)$ (resp. $J^-\tilde{\chi}(D_1/\kappa)$) will be bounded for any $\kappa > 0$. ($\tilde{\chi}$ was defined by (2.1).) To be more precise we consider the operators

$$\begin{aligned} &J^{\pm}\tilde{\chi}(\mp D_1/\kappa)^2(J^{\pm})^* u(x) - \tilde{\chi}(\mp D_1/\kappa)^2 u(x) \\ &= \iint e^{-i\phi^{\pm}(x,\xi)+i\phi^{\pm}(x',\xi)} a^{\pm}(x,\xi)\overline{a^{\pm}(x',\xi)}\tilde{\chi}(\mp\xi_1/\kappa)^2 u(x')dx'\,d\xi \\ &\quad - \iint e^{-i(x-x')\cdot\xi}\tilde{\chi}(\mp\xi_1/\kappa)^2 u(x')\,dx'\,d\xi \\ &= \iint e^{i(x'-x)\cdot\xi}\psi_0^{\pm}(x,\xi,x')u(x')dx'\,d\xi \end{aligned} \tag{5.6}$$

where the substitution

$$\tilde{\xi}(x,\xi,x') = \int_0^1 \nabla_x\phi^{\pm}(y + s(x-y),\xi)\,ds.$$

has been made and

$$\psi_0^{\pm}(x,\xi,x') = a^{\pm}(x,\tilde{\xi})\overline{a^{\pm}(x',\tilde{\xi})}\tilde{\chi}(\mp\tilde{\xi}_1/\kappa)^2\mathscr{J}(x,x',\xi) - \tilde{\chi}(\mp\xi_1/\kappa)^2 \tag{5.7}$$

where $\mathscr{J}$ is the Jacobian corresponding to the substitution. Therefore we have $J^{\pm}\tilde{\chi}(\mp D_1/\kappa)^2(J^{\pm})^* - \tilde{\chi}(\mp D_1/\kappa)^2 = \psi_0^{\pm}(X, D_x, X')$ is a pseudo-differential operator with symbol $\psi_0^{\pm} \in \mathscr{B}$ and with $L^2(\mathbb{R}^n)$ operator norm bounded by $C|\psi|_m$ for some constant C and integer m, by the Calderón–Vaillancourt theorem (Calderón and Vaillancourt, 1972). Moreover $|\psi_0^{\pm}|_m$ is small because of (2.9) and (3.6). In fact, given $r > 0$ we can choose $R > 0$ in (3.6) and $R_0 > 0$ in (2.9) so that $\|\psi_0^{\pm}(X, D_x, X')\| < r/2$. Adding we have

$$\|J^+\tilde{\chi}(-D_1/\kappa)^2(J^+)^* + J^-\tilde{\chi}(D_1/\kappa)^2(J^-)^* - \mathbf{1}\| < r \tag{5.8}$$

because $\tilde{\chi}(-\xi_1)^2 + \tilde{\chi}(\xi_1)^2 \equiv 1$. Later we shall see that $r > 0$ should be small to assure the convergence of a certain series but for now we just note that, provided $r < 1$, $J^+\tilde{\chi}(-D_1/\kappa)^2(J^+)^* + J^-\tilde{\chi}(D_1/\kappa)^2(J^-)^*$ is invertible.

The operators $Q^{\pm}$ defined by (5.1) can be shown to be bounded by a similar argument. In fact if the symbol $q^{\pm}$ is in $\mathscr{B}$ and supported on $\{\pm\Re\xi_1 < \kappa\}$ for some real κ then there are constants $C = C(\kappa) > 0$ and $m \in \mathbb{N}$ which do not depend on $q^{\pm}$ so that $\|Q^{\pm}\| \le C\|q^{\pm}\|_m$. See Dereziński and Gérard's book (1997, Theorem D.13.2) or White (2001, Equation 5.8).

Introduce a spectral representation for H_0. Let

$$G(\xi) = (1/3)\xi_1^3 + \xi_1(\xi_2^2 + \cdots + \xi_n^2)$$

and define the operator U on $L^2(\mathbb{R}^n)$ by

$$Uv(x) = F^{-n/2}[e^{iG(D)/F}v](x/F). \tag{5.9}$$

Then U is a unitary operator on $L^2(\mathbb{R}^n)$ and $H_0 = U^* x_1 U$. Now suppose $v \in \mathscr{S}(\mathbb{R}^n)$ and define, for each real λ, $T_0(\lambda)v(x_\perp) = Uv(\lambda, x_\perp)$, that is U, followed by restriction to the plane $x_1 = \lambda$ so that $T_0(\lambda)v$ is in $\mathscr{S}(\mathbb{R}^{n-1})$.

The following result is (White, 2001, Lemma 5.1) generalized to the long range case $\epsilon_V > 0$.

Lemma 1. *For any $k \in \mathbb{N}$ and $\kappa > 0$ there are constants $C > 0$ and $m \in \mathbb{N}$ so that for any operator $Q^\pm$ as in* (5.1) *with symbol $q^\pm$, where $q^\pm(x, \xi) = 0$ for $\pm\Re\xi_1 > \kappa$ for some $\kappa > 0$*

$$\|[\tilde{\chi}_{(0,\infty)} + \langle x_1\rangle^{-k}\tilde{\chi}_{(-\infty,0)}]Q^\pm[|x_1|^k\tilde{\chi}_{(-\infty,0)}]\| \le C|q^\pm|_m. \tag{5.10}$$

Here $\tilde{\chi}_{(0,\infty)}$ and $\tilde{\chi}_{(-\infty,0)}$ are as in (2.1). *In addition, for any real λ, the operator*

$$(\tilde{\chi}_{(0,\infty)} + \langle x_1\rangle^{-1}\tilde{\chi}_{(-\infty,0)})T_0(\lambda)^* \tag{5.11}$$

is bounded as a mapping from $L^2(\mathbb{R}^{n-1})$ to $L^2(\mathbb{R}^n)$ for every $\lambda \in \mathbb{R}$. In particular $[\tilde{\chi}_{(-\infty,0)} + \langle x_1\rangle^{-1}\tilde{\chi}_{(-\infty,0)}]QT_0(\lambda)^*$ is a bounded operator from $L^2(\mathbb{R}^{n-1})$ to $L^2(\mathbb{R}^n)$ whenever $|q|_m < \infty$.

Proof. The proof of Lemma 5.1 given in White (2001) is applicable here. There it was assumed that $\epsilon_V > 1/2$ but that hypothesis, we shall see, was unnecessary. A notational change is needed: our $Q^\pm$ and $q^\pm$ replace Q and q there; $\theta^\pm(x, \xi) = \phi^\pm(x, \xi) - x \cdot \xi$. The assumption $\epsilon_V > 1/2$ was used to conclude that that a certain operator was bounded on $L^2(\mathbb{R}^n)$. It was defined, for any $k \in \mathbb{N}$ by

$$u \mapsto \text{Os-}\iint e^{-ix'\cdot\xi}\sigma^\pm(x, \xi, x' + x)u(x' + x)\,dx'd\xi$$

where "Os-" indicates that the integral is an oscillatory integral (Kumano-go, 1981) and where $\tilde{\chi}$ was defined in (2.1) and

$$\sigma^\pm(x, \xi, x') = (-i)^k\tilde{\chi}(x_1)\frac{\partial^k}{\partial\xi_1^k}(e^{i\theta^\pm(x,\xi)}q^\pm(x, \xi))\left[\frac{x_1'}{x_1' - x_1}\right]^k\tilde{\chi}(-x_1' - 1)$$

However this operator can be rewritten in the same form as (5.1) with symbol $\tilde{q}^\pm$

$$\tilde{q}^\pm(x, \xi') = \text{Os-}\iint e^{-ix'\cdot\xi}\sigma^\pm(x, \xi + \xi', x + x' + y(x, \xi, \xi'))\,d_1x'\,d_1\xi$$

where $y(x, \xi, \xi') = \int_0^1 \nabla_\xi\theta^+(x, \xi' + s(\xi - \xi'))\,ds$ by changing the order of integration and making a change of variable. Since $\tilde{q}^\pm \in C_b^\infty(\mathbb{R}^n)$ the operator is bounded as noted above or see Dereziński and Gérard (1997, Theorem 13.2). $\square$

Next is the fundamental extension result.

Proposition 3.

(a) *Suppose $\mu > 0$ and $k \in \mathbb{N}_0$. Then there is $m = m(k)$ so that, for any operator $Q^\pm$ as in (5.1) with symbol $q^\pm \in \mathscr{B}$, such that $q^\pm(x, \xi) = 0$ for $\pm\Re\xi_1 > \kappa$ for some $\kappa > 0$ and $\|q^\pm\|_m < \infty$, the operators*

$$(-\Delta)^k h_\mu Q^\pm T_0(\lambda)^* \quad \text{and} \quad H_0^k h_\mu Q^\pm T_0(\lambda)^* \tag{5.12}$$

extend to entire functions of λ taking values in the space of bounded operators from $L^2(\mathbb{R}^{n-1})$ to $L^2(\mathbb{R}^n)$. Denote the extensions by the same symbols. For any compact $\mathscr{K} \subseteq \mathbb{C}$, there is a constant C not depending on $q^\pm$ so that, for all $\lambda \in \mathscr{K}$,

$$\|(-\Delta)^k h_\mu Q^\pm T_0(\lambda)^*\| + \|H_0^k h_\mu Q^\pm T_0(\lambda)^*\| < C\|q^\pm\|_m \tag{5.13}$$

(b) *For any $\nu > 0$, $e^{-\nu\langle D_1\rangle} T_0(\lambda)^*$ extends to $\{|\Im\lambda| < F\nu\}$ as an analytic operator valued function bounded from $L^2(\mathbb{R}^{n-1})$ to $L^2(\mathbb{R}^n)$.*

This is (White, 2001, Proposition 5.2) except that that result assumed $\epsilon_V > 1/2$ because it relied on White (2001, Lemma 5.1). However, we have just seen in Lemma 1 above that the assumption $\epsilon_V > 1/2$ in White (2001, Lemma 5.1) was superfluous and so we can omit it in Proposition 2 as well. Also Part (b) has been restated; it is an immediate consequence of White (2001, Equation (5.3)).

6. Theorem 2

Our next goal is to establish Theorem 2, assuming Theorem 1. We begin by taking a closer look at the effective potential $V_J^\pm$ discussed briefly at the beginning of §2. We consider V_J^+ to be specific or rather $Q^+ = V_J^+ - V_e J^+ \tilde{\chi}(-D_1/\kappa)$. Then Q^+ is an operator of the form (5.1) with symbol $h_\mu(t_1^+ + t_2^+) + t_3$ where, for $(x, \xi) \in \mathbb{R}^{2n}$,

$$\begin{aligned}
h_\mu t_1^+(x,\xi) &= t^+(x,\xi)\tilde{\chi}_{(-\infty,-2)}(\xi_1/\kappa_\sigma)\tilde{\chi}(-\xi_1/\kappa) \\
&\quad + iF(b(x,\xi) - 1)\tilde{\chi}(x_1 + R_1)[\tilde{\chi}_{(-\infty,-2)}(\xi_1/\kappa_\sigma)\tilde{\chi}(-\xi_1/\kappa)]' \\
&\quad + \tilde{\chi}(x_1 + R_1)p^+(x,\xi)(1 - \tilde{\chi}_{(-\infty,-2)}(\xi_1/\kappa_\sigma))\tilde{\chi}(-\xi_1/\kappa) \\
h_\mu t_2^+(x,\xi) &= iFb_e(x,\xi)[1 - \tilde{\chi}(x_1 + R_1)][\tilde{\chi}_{(-\infty,-2)}(\xi_1/\kappa_\sigma)\tilde{\chi}(-\xi_1/\kappa)]' \\
t_3^+(x,\xi) &= iF(b_{\mathscr{A}}(x,\xi) - 1)[1 - \tilde{\chi}(x_1 + R_1)][\tilde{\chi}_{(-\infty,-2)}(\xi_1/\kappa_\sigma)\tilde{\chi}(-\xi_1/\kappa)]' \\
&\quad + [1 - \tilde{\chi}(x_1 + R_1)]p^+(x,\xi)(1 - \tilde{\chi}_{(-\infty,-2)}(\xi_1/\kappa_\sigma))\tilde{\chi}(-\xi_1/\kappa) \\
&\quad - \frac{iF}{\kappa}\tilde{\chi}'(-\xi_1/\kappa)
\end{aligned}$$

for any $R_1 > 0$. (R_1 will be specified below.) Here t^+ is the symbol of Proposition 3.1 and $\tilde{\chi}_{(-\infty,-2)}$ was defined in (3.7) and p^+ was defined in (2.4). If $\mu > 0$ is suitably small then t_1^+ and t_2^+ are the symbols of bounded operators on $L^2(\mathbb{R}^n)$ which we denote V_1^+ and V_2^+; V_3^+ will be the (bounded) operator with symbol t_3^+ and so $V_J^+ = V_e J^+ \tilde{\chi}(-D_1/\kappa) + h_\mu(V_1^+ + V_2^+) + V_3$. As for V_J^-, similar reasoning applies. We define V_k^-, for $k = 1, 2, 3$ by complex conjugation $\overline{V_k^- u} = V_k^+ \bar{u}$ for any u in $\mathscr{S}(\mathbb{R}^n)$ say. Then we have

$$V_J^\pm = V_e J^\pm \tilde{\chi}(\mp D_1/\kappa) + h_\mu(V_1^\pm + V_2^\pm) + V_3^\pm. \tag{6.1}$$

The logic for this breakdown of $V_J^{\pm}$ will become apparent in the next section but briefly it is this: $V_1^{\pm}$ and $V_2^{\pm}$ are bounded operators from which we have been able to extract the exponential decay factor h_μ and moreover $V_1^{\pm}$ is H_0-compact and $\|V_2^{\pm}\| = o(1)$ as $R_1 \to \infty$. As for $V_3^{\pm}$, its symbol $t_3^{\pm}$ is in $\mathscr{B}_{\mathscr{A}}$ and it is supported in the strip $\{|\Re\xi_1| < \kappa\}$ and $\|V_3^{\pm}\|$ is small provided R_1 and κ are large enough.

Outline of the Proof of Theorem 2. We assume Theorem 1 and derive Theorem 2. The derivation of (1.6) is well known in the case $F = 0$ (Isozaki and Kitada, 1985a, Theorem 3.3) and in the case $F \neq 0$, the proof is similar (White, 2001, Theorem 1.1). We shall show that each of the terms on the right side of (1.6) has a meromorphic extension beginning with the term $T_0(\lambda)V_J^* R(\lambda + i0)V_J T_0(\lambda)^*$. Since $V_J = \sum_{\pm} V_J^{\pm}$ we will first show that $T_0(\lambda)(V_J^+)^* R(\lambda + i0)V_J^+ T_0(\lambda)^*$ has such an extension. We do this in steps by replacing each of the occurrences of V_J^+ by one of the terms on the right side of (6.1) and showing that that term can be continued. For instance

$$\begin{aligned} &T_0(\lambda)(V_1^+)^* h_\mu R(\lambda + i0) h_\mu V_1^+ T_0(\lambda)^* \\ &\quad = [T_0(\lambda)(V_1^+)^* e^{-\mu_1 x_1^-}] f [e^{-\mu_1 x_1^-} R(\lambda + i0) e^{-\mu_1 x_1^-}] f [e^{-\mu_1 x_1^-} V_1 T_0(\lambda)^*] \end{aligned}$$

where f is the operator of multiplication by $f(x_1) = h_\mu(x_1) e^{2\mu_1 x_1^-}$ and $0 < \mu_1 \leq \mu/2$ so that f is a bounded. Each of the bracketed factors on the right-hand side of the above equation extend into the lower half of the complex plane as bounded operators by Theorem 1 and Proposition 5.2. As a further instance, we consider

$$\begin{aligned} &T_0(\lambda)(V_3^+)^* R(\lambda + i0) V_3^+ T_0(\lambda)^* \\ &\quad = [T_0(\lambda) e^{-\nu\langle D_1\rangle}] \tilde{f}(D_1) [(V_3^+)^* R(\lambda + i0) V_3] \tilde{f}(D_1) [e^{-\nu\langle D_1\rangle} T_0(\lambda)^*] \end{aligned}$$

where $\nu > 0$ is arbitrary and $\tilde{f}(\xi_1) = e^{\nu\langle\xi_1\rangle}\chi_{[-\kappa,\kappa]}(\xi_1)$ where $\kappa > 0$ is large enough that the interval $-\kappa \leq \xi_1 \leq \kappa$ contains the support of the symbol $t_3^+(x, \xi)$ of V_3^+ and $\chi_{[-\kappa,\kappa]}$ is the characteristic function of that interval. Again each of the bracketed factors has a meromorphic extension to the lower complex plane. To check this observe that $V_3^* R(\lambda + i0) V_3$ has such an extension because $(V_J^+)^* R(\lambda + i0) V_J^+$ does, by Theorem 1 and because $V_J^+ - V_3^+ = e^{-\mu x_1^-} A$ for some bounded operator A by (6.1). Also $e^{-\nu\langle D_1\rangle} T_0(\lambda)^*$ extends to $\{|\Im\lambda| < F\nu\}$ by Proposition 2. It is now a routine matter to check the remaining cases and show that $T_0(\lambda)(V_J^+)^* R(\lambda + i0) V_J^+ T_0(\lambda)^*$ has a meromorphic extension from $\mathbb{C}_+$ to $\mathbb{C}$ as a bounded operator on $L^2(\mathbb{R}^{n-1})$. The argument when either occurrence of V_J^+ is replaced by V_J^- is similar.

It remains to check that $T_0(\lambda) J^* V_J T_0(\lambda)^*$ has an analytic extension. Since $V_J = \sum_{\pm} V_J^{\pm}$ and $J = \sum_{\pm} J^{\pm}\tilde{\chi}(\mp D_1/\kappa)$, we will actually show that the operator $T_0(\lambda)\tilde{\chi}(-D_1/\kappa)(J^+)^* V_J^+ T_0(\lambda)^*$ has an analytic extension; the other terms, where "−" replaces "+," can be treated similarly and so the result will follow. As in the preceding paragraph we shall replace V_J^+ by each of the terms on the right side of (6.1) and show that each term extends. For example, we consider

$$\begin{aligned} &T_0(\lambda)\tilde{\chi}(-D_1/\kappa)(J^+)^* V_e J^+ \tilde{\chi}(-D_1/\kappa) T_0(\lambda)^* \\ &\quad = [T_0(\lambda)\tilde{\chi}(-D_1/\kappa)(J^+)^* e^{-\mu_1 x_1^-}](f(x_1) V_0 R_0(i)) \\ &\quad\quad \times [(H_0 - i) e^{-\mu_1 x_1^-} J^+ \tilde{\chi}(-D_1/\kappa) T_0(\lambda)^*] \end{aligned}$$

with f as above and $0 < 2\mu_1 \leq \mu_V$ (μ_V as in the Hypotheses.) Each of the bracketed factors has an analytic extension by Proposition 3. As a further instance we consider

$$\begin{aligned} &T_0(\lambda)\tilde{\chi}(-D_1/\kappa)(J^+)^* V_3^+ T_0(\lambda)^* \\ &\quad = T_0(\lambda)\tilde{\chi}(-D_1/\kappa)(J^+)^* V_3^+ \tilde{f}(D_1)[e^{-\nu_1\langle D_1\rangle} T_0(\lambda)^*] \end{aligned}$$

with $\tilde{f}$ as above. Certainly $e^{-\nu_1\langle D_1\rangle} T_0(\lambda)^*$ extends analytically by Proposition 5.1. Therefore we wish to show that $T_0(\lambda)\tilde{\chi}(-D_1/\kappa)(J^+)^* V_3^+$ also extends analytically. To do this we recall that J^+ has symbol $a^+ = a_e^+ + a_{\mathscr{A}}^+$ where $a_{\mathscr{A}}^+ \in \mathscr{B}_{\mathscr{A}}$ and for some $\mu > 0$, $a_e^+/h_\mu \in \mathscr{B}$; see the example above. Correspondingly we can write

$$J^+ = h_\mu J_1^+ + J_{\mathscr{A}}^+ \tag{6.2}$$

where J_1^+ (resp. $J_{\mathscr{A}}^+$) has symbol a_e^+/h_μ (resp. $a_{\mathscr{A}}^+$). Therefore we must check that $T_0(\lambda)(J_1^+)^* h_\mu V_3^+ + T_0(\lambda)\tilde{\chi}(-D_1/\kappa)(J_{\mathscr{A}}^+)^* V_3^+$ extends analytically. The first term does by Proposition 5.2 Part (a). As for $T_0(\lambda)\tilde{\chi}(-D_1/\kappa)(J_{\mathscr{A}}^+)^* V_3^+$ since the symbol $t_3^+(x, \xi)$ of V_3^+ is compactly supported in the ξ_1 variable (and so decays exponentially fast in ξ_1) we should be able to apply Proposition 5.2 Part (b) but first a variant of a commutation argument which shows how the exponential decay "commutes to the left" is needed and that is given by White (2001, Proposition 5.3). That result uses the fact that V_3^+ and $J_{\mathscr{A}}^+$ have symbols in $\mathscr{B}_{\mathscr{A}}$. With this observation the reader can now easily complete the check that $T_0(\lambda) J^* V_J J^+ \tilde{\chi}(-D_1/\kappa) T_0(\lambda)^*$ extends analytically to all of $\mathbb{C}$ and this completes the Proof of Theorem 2. □

7. Theorem 1

In this section we prove Theorem 1. We begin by stating an analogue of Theorem 1 for the continuation of the free resolvent $R_0(z) = (H_0 - z)^{-1}$. It utilizes the exponential weight function $h_{\mu,X}(x_1)$ of (5.4) in and also the operator $h_{\mu,X}(-|D_1|)$ which is Fourier equivalent to multiplication by the function $h_{\mu,X}(-|\xi_1|)$ which is $e^{-\mu|\xi_1 - X|}$ if $|\xi_1| > X + 1$.

Proposition 4. *Suppose that $Q_k^\pm$, for $k = 1, 2$ are integral operators as in* (5.1) *with symbols $q_k^\pm$ in $\mathscr{B}$ and for some $\kappa_0 > 0$, $q_k^\pm(x, \xi) = 0$ if $\pm\Re\xi_1 > \kappa_0$. Let $\mu, \nu, \kappa > 0$ and $R \geq 0$ be arbitrary. Define P_1 to be either $P_1 = h_{\mu,R} Q_1^\pm$ or $P_1 = h_{\nu,\kappa}(-|D_1|)$ and similarly define $P_2 = h_{\mu,R} Q_2^\pm$ or $P_2 = h_{\nu,\kappa}(-|D_1|)$ (a total of nine possibilities for the pair (P_1, P_2)). Then, in any case, there exists $m \in \mathbb{N}$ so that whenever $\|q_1^\pm\|_m < \infty$ and $\|q_2^\pm\|_m < \infty$ then $P_1 R_0(z) P_2^*$ has an analytic extension from $\mathbb{C}_+$ (resp. $\mathbb{C}_-$) to $\{\Im z > -\nu\}$ (resp. $\{\Im z < \nu\}$) as a bounded operator on $L^2(\mathbf{R}^n)$. (If $P_k \neq h_{\nu,\kappa}(-|D_1|)$ for $k = 1$ and 2 both then the extension is to all of $\mathbb{C}$.) Moreover the extensions are bounded in operator norm uniformly for $\|q_1^\pm\|_m \leq 1$, $\|q_2^\pm\|_m \leq 1$ and locally uniformly in z, $\Im z > -\nu$ (resp. $\Im z < \nu$). The extension is denoted $P_1 R_{0,+}(z) P_2^*$ (resp. $P_1 R_{0,-}(z) P_2^*$). Additionally*

$$P_1 R_{0,+}(z) P_2^* = P_1 R_0(z) P_2^* + 2\pi i P_1 T_0(z)^* T_0(z) P_2^*$$

$$(\textit{resp. } P_1 R_{0,-}(z) P_2^* = P_1 R_0(z) P_2^* - 2\pi i P_1 T_0(z)^* T_0(z) P_2^*)$$

for $\Im z < 0$ (resp. $\Im z > 0$). If one further supposes that $Q_1^\pm$ is bounded as a mapping from the domain $D(H_0)$ of H_0 (with the graph norm) to itself then, in the cases that

$P_1 = h_{\mu,R} Q_1^{\pm}$, $P_1 R_{0,+}(z) P_2^*$ and $P_1 R_{0,-}(z) P_2^*$ are bounded from $L^2(\mathbf{R}^n)$ to $D(H_0)$ and $H_0 P_1 R_{0,+}(z) P_2^*$. Also $H_0 P_1 R_{0,-}(z) P_2^*)$ is bounded, as an operator on $L^2(\mathbf{R}^n)$, uniformly for $\|q_1^{\pm}\|_m \leq 1$, $\|q_2^{\pm}\|_m \leq 1$ and locally uniformly in z.

This is (White, 2001, Proposition 6.1) except the hypothesis $\epsilon_V > 1/2$ was needed there because that result relied on White (2001, Proposition 5.2). In view of Proposition 5.2 above, $\epsilon_V > 0$ suffices.

We shall now express the resolvent $R(z)$ in terms of $R_0(z)$ in preparation for the proof of Theorem 1. We will use a two Hilbert space version of the resolvent identity. Before stating the identity we recall from (6.2) that $J^+ = h_\mu J_1^+ + J_{\mathscr{A}}^+$. Correspondingly we have $J^- = h_\mu J_1^- + J_{\mathscr{A}}^-$ where J_1^- and $J_{\mathscr{A}}^-$ are most simply defined by complex conjugation $\overline{J_1^- u} = J_1^- \overline{u}$ and $\overline{J_{\mathscr{A}}^- u} = J_{\mathscr{A}}^- \overline{u}$ for any $uin\mathscr{S}(\mathbb{R}^n)$. The decomposition $J^{\pm} = h_\mu J_1^{\pm} + J_{\mathscr{A}}^{\pm}$ is not unique and we can further specify that $\|J_1^{\pm}\|$ is small because, if not then replace the symbol $a_e^{\pm}/h_\mu$ of J_1^{pm} by $(1 - \tilde{\chi}(x_1 + R_2))a_e^{\pm}/h_\mu$ for suitably large R_2 and since $\tilde{\chi}(x_1 + R_2)a_e^+$ belongs to $\mathscr{B}_{\mathscr{A}}$ it can be incorporated into the symbols $a_{\mathscr{A}}^{\pm}$ of $J_{\mathscr{A}}^{\pm}$. Here we are using the decay of $a_e^{\pm}/h_\mu$ as $x_1 \to -\infty$ which may require adjusting $\mu > 0$. How small should $\|J_1^{\pm}\|$ be? So small that A and $\widetilde{A}$ are invertible where

$$\widetilde{A} = \sum_{\pm} J^{\pm} \tilde{\chi}(\mp D_1/\kappa)^2 (J_{\mathscr{A}}^{\pm})^* \quad \text{and} \quad A = \sum_{\pm} J_{\mathscr{A}}^{\pm} \tilde{\chi}(\mp D_1/\kappa)^2 (J_{\mathscr{A}}^{\pm})^*.$$

Therefore if $\|J_1^{\pm}\|$ is small enough then we can assure that $\|\widetilde{A} - \mathbf{1}\| < 2r$ and $\|A - \mathbf{1}\| < 2r$ and $r > 0$ is the constant of (5.8). Recall that $r > 0$ is a small parameter, and precisely how small will be specified in the proof of Theorem 1 below. For now we specify $r < 1/2$ so that $\widetilde{A}$ and A are invertible on $L^2(\mathbb{R}^n)$.

The resolvent identity convenient here is

$$R(z)\widetilde{A} = \sum_{\pm} [J^{\pm} \tilde{\chi}(\mp D_1) - R(z) V_J^{\pm}] R_0(z) \tilde{\chi}(\mp D_1/\kappa)(J_{\mathscr{A}}^{\pm})^* \tag{7.1}$$

We shall want to solve this equation for $R(z)$ and so it would be natural to multiply by $\widetilde{A}^{-1}$ but for technical reasons it will be better to multiply by A^{-1}. We have $\widetilde{A}A^{-1} - \mathbf{1} = \sum_{\pm} h_\mu J_1^{\pm} \tilde{\chi}(\mp D_1/\kappa)(J_{\mathscr{A}}^{\pm})^* A^{-1}$. Define further the operator

$$B(z) = \widetilde{A}A^{-1} + \sum_{\pm} [h_\mu V_2^{\pm} + V_3^{\pm}] R_0(z) \tilde{\chi}(\mp D_1/\kappa)(J_{\mathscr{A}}^{\pm})^* A^{-1}$$

where $V_2^{\pm}$, $V_3^{\pm}$ were introduced in the expansion (6.1) for $V_J^{\pm}$. The operator valued function $B(z)$ is analytic and invertible at least if $|\Im z|$ is large enough. In equation (7.1) we collect terms involving $R(z)$ and apply $A^{-1}B(z)^{-1}$ on the right to get

$$R(z)[\mathbf{1} + K(z)] = \sum_{\pm} J^{\pm} \tilde{\chi}(\mp D_1/\kappa) R_0(z) \tilde{\chi}(\mp D_1/\kappa)(J_{\mathscr{A}}^{\pm})^* A^{-1} B(z)^{-1} \tag{7.2}$$

where

$$K(z) = \sum_{\pm} [V_e J^{\pm} \tilde{\chi}(\mp D_1/\kappa) + h_\mu V_1^{\pm}] R_0(z) \tilde{\chi}(\mp D_1/\kappa)(J_{\mathscr{A}}^{\pm})^* A^{-1} B(z)^{-1}$$

and this is the starting point of the proof of Theorem 1.

Proof of Theorem 1. The idea behind the proof of Theorem 1 is quite simple. We consider $e^{-\mu x_1^-/2} R(z) e^{-\mu x_1^-/2}$, to be specific and we must show it has a meromorphic extension across the real axis. Equivalently we shall show $h_{\mu/2} R(z) h_{\mu/2}$ has such an extension. In equation (7.2) we multiply on the left and right by $h_{\mu/2}$ and then we show that the right hand side the resultant equation extends analytically across the real axis using Proposition 7.1. The left-hand side becomes $h_{\mu/2} R(z) h_{\mu/2}[\mathbf{1} + (1/h_{\mu/2}) K(z) h_{\mu/2}]$ and so if we show that $(1/h_{\mu/2}) K(z) h_{\mu/2}$ is an analytic compact operator valued function then Theorem 1 follows by applying the "analytic Fredholm" theorem (Kato, 1980, Theorem VII.1.9) or Reed and Simon (1980, Theorem VI.14) which says that if $\widetilde{K}(z)$ is an analytic, compact operator valued function defined on a connected domain and if $(\mathbf{1} + \widetilde{K}(z))^{-1}$ exists for some z then $(\mathbf{1} + \widetilde{K}(z))^{-1}$ is meromorphic. The obstacle to completing this proof is that h_μ does not commute with the two operators A^{-1} and $B(z)^{-1}$ so that Theorem 7.1 is not immediately applicable. Because of this we shall need an auxiliary result which is akin to a commutation result for $h_{\mu,R}$ and $h_{\nu,\kappa}(-|D_1|)$ with, not A^{-1}, but powers of $\mathbf{1} - A$. Of course $A^{-1} = \sum_{\ell=0}^{\infty} (\mathbf{1} - A)^\ell$ (Neumann series).

Lemma 2. *Suppose $Q_1^\pm$ and $Q_2^\pm$ are operators* (5.1) *with symbols $q_1^\pm$ and $q_2^\pm$ in $\mathscr{B}_{\mathscr{A}}$ with $q_k^\pm(x, \xi) = 0$ if $\pm\Re\xi_1 < \kappa_0$ for some real constant κ_0 and for $k = 1, 2$. Suppose $\ell \in \mathbb{N}_0$ and $R, \kappa, \mu, \nu > 0$ and $R, \kappa, 1/\mu$ are sufficiently large. Define $T^\pm = (Q_1^\pm)^* (\mathbf{1} - A)^\ell Q_2^\pm h_{\nu,\kappa}(-|D_1|)$ (resp. $T^\pm = (Q_1^\pm)^* (\mathbf{1} - A)^\ell h_{\mu,R}$). Then there exist $Q_k^\pm$, $3 \le k \le 6^{\ell+1}$ with symbols $q_k^\pm$ in $\mathscr{B}_{\mathscr{A}}$ such that $q_k^\pm(x, \xi) = 0$ if $\pm\Re\xi_1 < \kappa_0$ and operators $S_k^\pm$, $k \ge 2$ bounded on $L^2(\mathbb{R}^n)$ so that*

$$T^\pm = h_{\nu,\kappa}(-|D_1|) S_2^\pm + \sum_{k \ge 3} (Q_k^\pm)^* h_{\mu,R} S_k^\pm$$

Moreover there is $C > 0$ (not depending on $q_1^\pm$, $q_2^\pm$ or ℓ) so that $\|q_k^\pm\|_m < C \|q_1^\pm\|_m$ for $k \ge 3$ and all $m \in \mathbb{N}_0$ and there is $m_0 \in \mathbb{N}$ (again not depending on $q_1^\pm$ or $q_2^\pm$ or ℓ) so that $\|S_k\| < C^{\ell+1} r^\ell \|q_2^\pm\|_{m_0}$, for all $2 \le k \le 6^{k+1}$ (resp. $\|S_k\| < C^{\ell+1} r^\ell$) where $r > 0$ is the constant (5.8). *Finally, the order of "$\pm$" is immaterial: the statement remains true if $T^\pm = (Q_1^\mp)^* (\mathbf{1} - A)^\ell Q_2^\pm h_{\nu,\kappa}(-|D_1|)$.*

The motivation for this lemma is that it allows "commuting to the left" the exponential decay represented by the operators $h_{\mu,R}$ and $h_{\nu,\kappa}(-|D_1|)$. Leaving aside the proof of the lemma until §8 we shall show how it implies Theorem 1.

The first thing to note is that the lemma implies a certain extension of Proposition 1 which can be stated as follows. Let $Q_2^\pm$, and $Q_3^\pm$ be two operators as in (5.1) with symbols $q_2^\pm, q_3^\pm \in \mathscr{B}_{\mathscr{A}}$ which are supported in a region $\pm\Re\xi_1 < \kappa_0$ for some $\kappa_0 > 0$. Let P_1, q_1 $\mu, \nu, \kappa > 0$ and $R \ge 0$ be as in Proposition 1. Then, there is $m \in \mathbb{N}_0$, so that, if $\|q_k^\pm\|_m \le 1$ for $k = 1, 2, 3$ then

1. $P_1 R_0(z) (Q_2^\pm)^* A^{-1} h_{\mu,R}$
2. $P_1 R_0(z) (Q_2^\pm)^* A^{-1} Q_3^\pm h_{\nu,\kappa}(-|D_1|)$
3. $P_1 R_0(z) (Q_2^\mp)^* A^{-1} Q_3^\pm h_{\nu,\kappa}(-|D_1|)$

extend from $\mathbb{C}_+$ (resp. $\mathbb{C}_-$) to $\{\Im z > -\nu\}$ (resp. $\{\Im z < \nu\}$) as operators bounded uniformly for $\|q_k^\pm\|_m \le 1$, $k = 1, 2, 3$ and locally uniformly in z. (We can assume $\nu = \infty$ in case 1.) To verify this one expands $A^{-1} = \sum_{\ell \ge 0} (\mathbf{1} - A)^\ell$ and applies Lemma 2 to every summand. The ℓth summand becomes at most $6^{\ell+1}$ terms after the application of Lemma 2. To each of these $6^{\ell+1}$ terms, Proposition 4 applies to show that each term extends in z across the real axis. Moreover, if z is restricted to a compact subset of $\{\Im z > -\nu\}$ (resp. $\{\Im z > -\nu\}$) then the extension is bounded in operator norm by $C^{\ell+1} r^{\ell} \|q_1\|_m \|q_2\|_m \|q_3\|_m$ for some constants $C > 0$ and $m \in \mathbb{N}$ not depending on ℓ. It follows that, if r in (5.8) is chosen so that $r < 1/6C$ then the extensions form a uniformly convergent series and so the sum is analytic and is the required extension. (The choice of r will depend on the compact subset of $\mathbb{C} \ni z$.)

It now follows that $(\mathbf{1} - B(z))h_{\mu,R}$ and $(\mathbf{1} - B(z))Q_3^{\pm} h_{\nu,\kappa}(-|D_1|)$ have analytic extensions across the real axis. Indeed each of the operators

1. $P_1 R_0(z)(Q_2^{\pm})^* A^{-1}(\mathbf{1} - B(z))^{\ell} h_{\mu,R}$
2. $P_1 R_0(z)(Q_2^{\pm})^* A^{-1}(\mathbf{1} - B(z))^{\ell} Q_3^{\pm} h_{\nu,\kappa}(-|D_1|)$
3. $P_1 R_0(z)(Q_2^{\mp})^* A^{-1}(\mathbf{1} - B(z))^{\ell} Q_3^{\pm} h_{\nu,\kappa}(-|D_1|)$

also has such an extension at least in the case $\ell = 1$, for recall that $\widetilde{A} A^{-1} - \mathbf{1} = \sum_{\pm} h_\mu J_1^{\pm} \tilde{\chi}(\mp D_1/\kappa)(J_{\mathscr{A}}^{\pm})^* A^{-1}$.. (See the discussion after equation 7.1.) The same is true for general $\ell \in \mathbb{N}_0$ by analogous reasoning. Next we would like to conclude that, if one replaces $(\mathbf{1} - B(z))^{\ell}$ by $B(z)^{-1}$ in the above three expressions then the new expression also have analytic continuations. This follows by summing over ℓ because $B(z)^{-1} = \sum_{\ell \geq 0}(1 - B(z))^{\ell}$. The series is summable provided that $\|J_1\|$, $\|V_2\|$, and $\|V_3\|$ are chosen small enough (by choosing $R_1, R_2 > 0$ and $\kappa > 0$ large enough in their definitions) and provided that z is restricted to a compact subset of $\mathbb{C}$.

We are now ready to show that $h_{\mu/2} R(z) h_{\mu/2}$ has a meromorphic extension. Multiply on the left and right of equation (7.2) by $h_{\mu/2}$:

$$\begin{aligned} &h_{\mu/2} R(z) h_{\mu/2}[\mathbf{1} + (1/h_{\mu/2}) K(z) h_{\mu/2}] \\ &\quad = \sum_{\pm} h_{\mu/2} J^{\pm} \tilde{\chi}(\mp D_1/\kappa) R_0(z) \tilde{\chi}(\mp D_1/\kappa)(J_{\mathscr{A}}^{\pm})^* A^{-1} B(z)^{-1} h_{\mu/2} \end{aligned}$$

We have just seen that the right side of the above equation has an analytic extension across the real axis and so does $(1/h_{\mu/2}) K(z) h_{\mu/2}$. (We recall $\mu/2 < \mu_V$.) Next we show $(1/h_{\mu/2}) K(z) h_{\mu/2}$ is compact. Certainly $V_1^{\pm}$ is compact because its symbol $t_1^{\pm}(x, \xi) \to 0$ along with its derivatives as $|(x, \xi)| \to \infty$. Also $V_e R_0(i)$ is compact by the Hypotheses so that $V_e(H_{\mathscr{A}} - i)^{-1}$ is compact since the domains of H_0 and $H_{\mathscr{A}}$ are the same. Therefore we wish to show that

$$J^{\pm} \tilde{\chi}(\mp D_1/\kappa) R_0(z) \tilde{\chi}(\mp D_1/\kappa)(J_{\mathscr{A}}^{\pm})^* A^{-1} B(z)^{-1} h_{\mu/2}$$

is bounded from $L^2(\mathbb{R}^n)$ to $D(H_0)$ and by Proposition 4 it suffices to check that $J^{\pm} \tilde{\chi}(\mp D_1/\kappa)$ is a bounded operator on $D(H_0)$. That follows because $H_{\mathscr{A}} J^{\pm} \tilde{\chi}(\mp D_1/\kappa) - J^{\pm} \tilde{\chi}(\mp D_1/\kappa) H_0$ is bounded on $L^2(\mathbb{R}^n)$ (by Proposition 2). It now follows that $h_{\mu/2} R(z) h_{\mu/2}$ has a meromorphic extension by the analytic Fredholm theorem as indicated in the opening paragraph of this proof. Since the extension exists for some $\mu > 0$ it exists far all $\mu > 0$.

The second case we consider is that of $h_\mu R(z) V_3^{\pm}$. In equation (7.2) we multiply on the left by h_μ and right by $V_3^{\pm}$:

$$\begin{aligned} &h_\mu R(z) V_3^{\pm} \\ &\quad = -h_\mu R(z) K(z) V_3^{\pm} + \sum_{\pm} h_\mu J^{\pm} \tilde{\chi}(\mp D_1/\kappa) R_0(z) \tilde{\chi}(\mp D_1/\kappa)(J_{\mathscr{A}}^{\pm})^* A^{-1} B(z)^{-1} V_3^{\pm} \end{aligned}$$

The sum on the right-hand side has an analytic extension because $V_3^\pm = V_3^\pm h_{\nu,\kappa}(-|D_1|)$ for $\kappa > 0$ large enough and arbitrary $\nu > 0$ (because the symbol t_3 is supported in the strip $-\kappa < \xi_1 < \kappa$. The first term on the right-hand side has an extension by the first case already established. Therefore $(V_3^\mp)^* R(z) h_\mu$ has a meromorphic extension. This, in turn, implies that $(V^\pm)^* R(z) h_\mu$ has a meromorphic extension because $(1/h_\mu)(V^\pm - V_3^\pm)$ is bounded.

The proof that $(V_3^\pm)^* R(z) h_\mu$ has a meromorphic extension is similar but requires an additional argument: we multiply (7.2) on the left by $(V_3^\pm)^*$ and the right by $h_{\mu/2}$. When checking that the right-hand side of the resultant equation has an analytic extension we should recall (6.2): $J^\pm = h_\mu J_1^\pm + J_{\mathscr{A}}^\pm$. The expression involving $(V_3^\pm)^* J_{\mathscr{A}} = h_{\nu,\kappa}(-|D_1|)(V_3^\pm)^* J_{\mathscr{A}}$ can be seen to extend if we apply White (2001, Proposition 5.3b) (which requires the analyticity of the symbol of $J_{\mathscr{A}}$).

It can be shown that $(V_3^\pm)^* R(z) V_3^\pm$ and $(V_3^\mp)^* R(z) V_3^\pm$ have meromorphic extensions by the same arguments as in the previous two cases. We can interchange $V^\pm$ and $V_3^\pm$ in any of the five cases because $(1/h_\mu)(V^\pm - V_3^\pm)$ is bounded. The five cases together imply Theorem 1 except that Lemma 2 has yet to be verified. □

8. Lemma 7.2

In this section we give a proof of Lemma 7.2 and thereby complete the proof of Theorem 1. Of interest in the proof is that the analyticity plays a key role allowing a change of path of integration to obtain the necessary estimates.

Proof of Lemma 7.2. Let us express $\mathbf{1} - A = \sum_\pm \tilde{\chi}(\mp D_1/\kappa)^2 - A^\pm$ as a pseudo-differential operator. We recall that $A^\pm = J_{\mathscr{A}}^\pm \tilde{\chi}(\mp D_1/\kappa)^2 (J_{\mathscr{A}}^\pm)^*$ and repeat the computation (5.6) in the present setting (replace $J^\pm$ there by $J_{\mathscr{A}}^\pm$). We find that $A^\pm - \tilde{\chi}(\mp D_1/\kappa)^2$ is a pseudo-differential operator with symbol $\psi_{\mathscr{A}}^\pm$ say where $\psi_{\mathscr{A}}^\pm$ is given by equation (5.7) if one replaces $\psi_0^\pm$ there by $\psi_{\mathscr{A}}^\pm$ and $a^\pm$ by $a_{\mathscr{A}}^\pm$. Then we have $\mathbf{1} - A = -\sum_\pm \psi_{\mathscr{A}}^\pm(X, D_x, X')$ and $\psi_{\mathscr{A}}^\pm \in \mathscr{B}_{\mathscr{A}}$ and $\|\psi_{\mathscr{A}}^\pm(X, D_x, X')\| < 3r$ where $r > 0$ was defined in (5.8) by an argument given near (7.1).

The proof of Lemma 7.2 is by induction on ℓ. It may seem that this is unnecessary since $(1 - A)^\ell$ is itself just a pseudo-differential operator. However the analytic properties of the symbol of $1 - A$ are not clearly reflected in the symbol of the composite operator $(1 - A)^\ell$. Let us indicate the basic ingredients of the induction argument. One involves considering $\psi_1(X, D_x, X')h_{\mu,R}$ when $\psi_1 \in \mathscr{B}_{\mathscr{A}}$; $\psi_1 = \psi_{\mathscr{A}}^\pm$ is the archetypical example.

We wish to show that $\psi_1(X, D_x, X')h_{\mu,R}$ can be written in the form

$$\psi_1(X, D_x, X')h_{\mu,R} = \psi_{\nu,\kappa}(X, D_x, X') + h_{\mu,R} S u(x) \tag{8.1}$$

where S is a bounded operator and $\psi_{\nu,\kappa}(X, D_x, X')$ is the pseudo-differential operator with symbol $(x, \xi, x') \mapsto h_{\nu,\kappa}(-|\xi_1|)\psi(x, \xi, x')$ for some $\psi \in \mathscr{B}_{\mathscr{A}}$. Moreover, there are absolute constants $C > 0$ and $m \in \mathbb{N}$ so that $\|S\| < C\|\psi_1\|_m$. This is analogous to a commutator computation: the exponential decay factor $h_{\mu,R}$ on the left side of (8.1) gives rise to two terms, one where the $h_{\mu,R}$ multiplies on the left and the other with exponential decay $h_{\nu,\kappa}(-|\xi_1|)$ in ξ_1. In both terms the decay has "moved to the left." Whatever is to the right of the exponential decay becomes uninteresting and we need only record that it is a bounded operator, in this case S.

We now prove (8.1). Since $\psi_1 = [(1 - h_{\mu,R}) + h_{\mu,R}]\psi_1$ we see that it suffices to verify (8.1) with ψ_1 replaced by $(1 - h_{\mu,R})\psi_1$ because $\psi_1(X, D_x, X')$ is itself bounded by the Calderón–Vaillancourt theorem (Calderón and Vaillancourt, 1972). Therefore we may as well suppose that $\psi_1(x, \xi, x')$ is supported in the region where $1 - h_{\mu,R}(x_1)$ is, that is where $x_1 < -R + 1$. Suppose that $u \in \mathscr{S}(\mathbb{R}^n)$ and that μ, R, and κ_1 are positive parameters so that $1/\mu, R, \kappa_1$ are large. By a change of the path of integration

$$[\psi_1(X, D_x, X')h_{\mu,R}]u(x) \\ = \iint e^{i(x-x')\cdot\xi} e^{\zeta(\xi_1)(x_1-x_1')}\psi_1(x, \xi - i\zeta(\xi_1), x')h_{\mu,R}(x_1')u(x')d_1x'd_1\xi$$

where ζ is a smooth positive function which is 0 if $|\xi_1| < \kappa_1$ and is μ if $\xi_1 > \kappa_1 + 1$. Provided $1/\mu, \kappa_1$ are large enough, the path is indeed in the domain of analyticity of $\psi_1 \in \mathscr{B}_{\mathscr{A}}$. Therefore,

$$[\psi_1(X, D_x, X')h_{\mu,R}]u(x) \\ = \iint e^{i(x-x')\cdot\xi}\big[e^{\zeta(\xi_1)(x_1-x_1')}h_{\nu,\kappa}(-|\xi_1|)\psi_1(x, \xi - i\zeta(\xi_1), x') \\ + e^{\mu(x_1-x_1')}(1 - h_{\nu,\kappa}(-|\xi_1|))\psi_1(x, \xi - i\mu, x')\big]h_{\mu,R}(x_1')u(x')d_1x'd_1\xi$$

where $\nu > 0$ is arbitrary and $\kappa > 0$ is so large that $1 - h_{\nu,\kappa}(-|\xi_1|) \neq 0$ implies $\zeta(\xi_1) = \mu$: $\kappa > \kappa_1 + 2$ suffices. This verifies (8.1) if we define

$$\psi(x, \xi, x') = e^{\zeta(\xi_1)(x_1-x_1')}\psi_1(x, \xi - i\zeta(\xi_1), x')h_{\mu,R}(x_1').$$

so that $\psi \in \mathscr{B}_{\mathscr{A}}$. To check that S in (8.1) satisfies $\|S\| < C\|\psi_1\|_m$ for absolute constants $C > 0$ and $m \in \mathbb{N}$ we need only apply the Calderón–Vaillancourt theorem (Calderón and Vaillancourt, 1972). Later in the proof we apply (8.1) when $\psi_1 = \psi_{\mathscr{A}}^{\pm}$ and we will be able to conclude $\|S\| < Cr$.

Equation (8.1) is one ingredient of the induction argument; a second ingredient is as follows. Suppose $\psi_1, \psi_2 \in \mathscr{B}_{\mathscr{A}}$. Then for any $\nu > 0$ and all μ, κ, and R so that $1/\mu, \kappa, R > 0$ are large enough there exist symbols $\psi_k \in \mathscr{B}_{\mathscr{A}}$ and bounded operators S_k, $3 \leq k \leq 7$ so that

$$\psi_1(X, D_x, X')\psi_{2,\nu,\kappa}(X', D_{x'}, X'') \\ = \sum_{k=3,4,5} \psi_{k,\nu,\kappa}(X, D_x, X')S_k + \sum_{k=6,7} \psi_k(X, D_x, X')h_{\mu,R}S_k \tag{8.2}$$

Moreover there exist constants $C_m, C > 0$ and $m_0 \in \mathbb{N}$ not depending on ψ_1, ψ_2 so that $\|\psi_k\|_m \leq C_m\|\psi_1\|_m$ for all $m \in \mathbb{N}_0$ and $\|S_k\| < C\|\psi_2\|_{m_0}$ for $3 \leq k \leq 7$. Here the notation $\psi_{k,\nu,\kappa}(X, D_x, X')$ refers to the pseudo-differential operators with symbol $(x, \xi, x') \mapsto h_{\nu,\kappa}(-|\xi_1|)\psi_k(x, \xi, x')$, for $k = 2, 3, 4, 5$. We observe that $h_{\nu,\kappa}(-|\xi_1|)\psi_1(x, \xi, x')\psi_{2,\nu,\kappa}(X', D_{x'}, X'')$ is already of the same form as the first term on the right side of (8.2) (with $S_3 = \psi_{2,\nu,\kappa}(X', D_{x'}, X''))$ and so it suffices to look for an expansion like (8.2) when $[1 - h_{\nu,\kappa}(-|\xi_1|)]\psi_1(x, \xi, x')$ replaces $\psi_1(x, \xi, x')$ there. Thus we may assume that $\psi_1(x, \xi, x') = 0$ if $|\xi_1| < \kappa - 1$; we shall in fact assume that $\psi_1(x, \xi, x') = 0$ if $\xi_1 < \kappa - 1$ since the other case $\xi_1 > -\kappa + 1$ is very similar.

We suppose $u \in \mathcal{S}(\mathbb{R}^n)$ and make a change of path of integration

$$\begin{aligned}&\psi_1(X, D_x, X')\psi_{2,\nu,\kappa}(X', D_{x'}, X'')u(x)\\&= \iiiint e^{i(x-x')\cdot\xi}e^{i(x'-x'')\cdot\xi'}e^{-\omega(x_1')(\xi_1-\xi_1')}\psi_1(x, \xi, x' - i\omega(x_1')\mathbf{e}_1)\\&\quad\times \psi_2(x' - i\omega(x_1')\mathbf{e}_1, \xi', x'')h_{\nu,\kappa}(-|\xi_1'|)u(x'')d_1x''d_1\xi'd_1x'd_1\xi\end{aligned}$$

where ω is a smooth nonnegative function which is $\omega(x_1) = 0$ if $x_1 > -R'$ but is the constant $\omega(x_1) = \nu$ if $x_1 < -R' - 1$. Provided R' is large enough the path is in the domain of analyticity of $\psi_1, \psi_2 \in \mathcal{B}_{\mathcal{A}}$. (Each of the four iterated integrals exists absolutely provided that the integration is carried out in the prescribed order $dx'' \to d\xi' \to dx' \to d\xi$.) Introduce into the integrand $1 = h_{\mu,R}(x') + (1 - h_{\mu,R}(x'))$ where $R > R' + 2$. The term corresponding to $h_{\mu,R}$ can be expressed as $\psi_6(X, D_x, X')h_{\mu,R}S_6$ (as in (8.2)). The other term containing $1 - h_{\mu,R}(x')$ is supported where $\omega(x_1) = \nu$ by the choice of R and we have $e^{-\nu(\xi_1-\xi_1')}h_{\nu,\kappa}(-|\xi_1'|) < C_\nu e^{-\nu(\xi_1-\kappa)}$ when $\xi_1 > \kappa - 1$. Since ψ_1 is supported in the region $\xi_1 > \kappa - 1$ this other term is of the form $\psi_{4,\nu,\kappa}(X, D_x, X')\widetilde{S}_4$ and this establishes (8.2). We observe that in the case that $\psi_1(x, \xi, x')$ is supported on the region $\xi_1 < -\kappa + 1$ then the path of integration should be $x' + \omega(x_1')i$ as opposed to $x' - \omega(x_1')i$.

The statement (8.2) remains valid if $Q_2^{\pm}$ replaces $\psi_2(X', D_{x'}, X'')$ where $Q_2^{\pm}$ is as in the statement of Lemma 7.2. More precisely, if $\psi_1 \in \mathcal{B}_{\mathcal{A}}$ then for any $\nu > 0$ and all μ, κ, and R so that $1/\mu, \kappa, R > 0$ are large enough there exist symbols $\tilde{\psi}_k \in \mathcal{B}_{\mathcal{A}}$ and bounded operators $\widetilde{S}_k$, $3 \le k \le 7$ so that

$$\begin{aligned}&\psi_1(X, D_x, X')Q_2^{+}h_{\nu,\kappa}(-|D_1|)\\&= \sum_{k=3,4,5} \tilde{\psi}_{k,\nu,\kappa}(X, D_x, X')\widetilde{S}_k + \sum_{k=6,7} \tilde{\psi}_k(X, D_x, X')h_{\mu,R}\widetilde{S}_k\end{aligned} \tag{8.3}$$

Moreover there exist constants $C_m, C > 0$ and $m_0 \in \mathbb{N}$ not depending on ψ_1, q_2^{+} so that $\|\tilde{\psi}_k\|_m \le C_m\|\psi_1\|_m$ for all $m \in \mathbb{N}_0$ and $\|\widetilde{S}_k\| < C\|q_2^{+}\|_{m_0}$ for $3 \le k \le 7$. Indeed the proof of (8.3) is almost identical to (8.2). A similar statement is valid if Q_2^{+} and q_2^{+} are replaced by Q_2^{-} and q_2^{-} respectively.

The final ingredient needed for the induction argument is this: if $\psi_2 \in \mathcal{B}_{\mathcal{A}}$ then for any $\nu > 0$ and all μ, κ, and R so that $1/\mu, \kappa, R > 0$ are large enough there exist operators $Q_k^{\pm}$ (as in (5.1)) with symbols $q_k^{\pm} \in \mathcal{B}_{\mathcal{A}}$ such that $q_k^{\pm}(x, \xi) = 0$ if $\pm\Re\xi_1 < \kappa_0$ and bounded operators $\widetilde{S}_k^{\pm}$, $8 \le k \le 10$ so that

$$(Q_1^{\pm})^*\psi_{2,\nu,\kappa}(X, D_x, X') = h_{\nu,\kappa}(-|D_1|)\widetilde{S}_8^{\pm} + \sum_{k=9,10} (Q_k^{\pm})^*h_{\mu,R}\widetilde{S}_k^{\pm} \tag{8.4}$$

Moreover there exist constants $C_m, C > 0$ and $m_0 \in \mathbb{N}$ not depending on $\psi_2, q_1^{\pm}$ so that $\|q_k^{\pm}\|_m \le C_m\|q_1^{\pm}\|_m$ for all $m \in \mathbb{N}_0$ and $\|\widetilde{S}_k^{\pm}\| < C\|\psi_2\|_{m_0}$ for $8 \le k \le 10$. The proof of this statement is much the same as that of (8.3).

We now indicate how the expansion for $T^{\pm} = (Q_1^{\pm})^*(\mathbf{1} - A)^{\ell}Q_2^{\pm}h_{\nu,\kappa}(-|D_1|)$ can be derived from (8.1)–(8.4). This can be formalized by induction on ℓ. We have already seen that $A - \mathbf{1}$ is a pseudo-differential operator with symbol $\psi_{\mathcal{A}}^{+} + \psi_{\mathcal{A}}^{-}$ in $\mathcal{B}_{\mathcal{A}}$. We apply (8.3) to $(\mathbf{1} - A)Q_2^{\pm}h_{\nu,\kappa}(-|D_1|)$ which gives two terms with decay given by $h_{\mu,R}(x_1)$ and three terms with $h_{\nu,\kappa}(-|D_1|)$. To the former (8.1) applies and to the latter (8.2) applies. One continues to apply (8.1) or (8.2), whichever is applicable, to each of the terms generated. This process moves the exponential decay factors $h_{\mu,R}(x_1)$, and $h_{\nu,\kappa}(-|D_1|)$ to the left of $(\mathbf{1}-A)^{\ell}$ and at that stage (8.4) will apply to about half the terms and the result is the expansion claimed for $T^{\pm}$. The number of terms is at most $6^{\ell+1}$ approximately. (Significantly the number of terms does not grow faster than Ca^{ℓ} for some absolute constants $C > 0$ and $a > 1$). The final statement of Lemma 7.2 is established similarly. $\square$

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