4.4 $A = \mathbb{C}\mathfrak{S}_d$ $\quad S^\lambda = Ac_\lambda = Ap_\lambda \kappa_\lambda$

a. Define $\varphi : Ap_\lambda\kappa_\lambda \to A\kappa_\lambda p_\lambda$ by $\varphi(u) = up_\lambda$.
Then $\varphi(\sigma u) = \sigma u p_\lambda = \sigma \varphi(u)$ so
$\varphi$ is a $\mathbb{C}\mathfrak{S}_d$-map
Similarly let $\psi : A\kappa_\lambda p_\lambda \to Ap_\lambda\kappa_\lambda$ be $\psi(v) = v\kappa_\lambda$.

Then $\psi \circ \varphi(up_\lambda\kappa_\lambda) = uc_\lambda c_\lambda = n_\lambda \cdot uc_\lambda$.

Then $\psi\circ\varphi$ is mult by a nonzero scalar.
Hence $\varphi$ $\psi$ are $\cong$'s since $S^\lambda$ is irreducible

b. ~~This shows that the map $Ap_\lambda \to A$~~

~~De~~ Let $\varphi : Ap_\lambda \to A\kappa_\lambda$ be given by
$\varphi(u) = u\kappa_\lambda$ $\qquad S^\lambda = A\kappa_t p_t$

Clearly the image of $\varphi$ is $Ap_\lambda\kappa_\lambda \cong S^\lambda$ $\qquad S^{\lambda'} = Ap_{t'}\kappa_t$

c. Let $t$ be a $\lambda$-tableau and $t'$ be the transpose.

Ex $t = \begin{smallmatrix}1&2&3\\4&5\end{smallmatrix}$ $\quad t' = \begin{smallmatrix}1&4\\2&5\\3\end{smallmatrix}$

so $C_t = R_{t'}$ $\quad R_t = C_{t'}$ $\qquad \kappa_{t'} = \sum_{\sigma\in R_t} \operatorname{sgn}\sigma$

$S^\lambda \cong \mathbb{C}\mathfrak{S}_d p_t \kappa_t$ $\qquad \sigma p_t \kappa_t$ $\qquad p_{t'} = \sum_{\sigma \in C_t} \sigma$

$S^{\lambda'} \cong \mathbb{C}\mathfrak{S}_d \kappa_{t'} p_{t'}$

4.6 Let $\lambda = (d-1,1)$. Set $t = \begin{smallmatrix} 1 & 2 & 3 & \cdots & d-1 \\ d \end{smallmatrix}$, so $R_t = \Sigma_{d-1}$.

Then $\mathbb{C}\Sigma_d e_t$ is spanned by left coset sums of $R_t$, and cosets of $R_t$ are determined by where $d$ maps to. Thus

$$\text{Let } W_j = \sum_{\substack{\sigma \in \Sigma_d \\ \sigma(d)=j}} \sigma \qquad j = 1, 2, \ldots, d.$$

So $S^{\lambda} = \mathbb{C}\Sigma_d e_t k_t$ is spanned by $\{V_j\}_{j=1}^{d}$ where

$$V_j = W_j k_t = W_j(e - (1d))$$

$$V_j = \sum_{\substack{\sigma \in \Sigma_d \\ \sigma(d)=j}} \sigma - \sum_{\substack{\tau \in \Sigma_d \\ \tau(1)=j}} \tau .$$

First notice that $\displaystyle\sum_{j=1}^{d} V_j = \sum_{\sigma \in \Sigma_d} \sigma - \sum_{\tau \in \Sigma_d} \tau = 0.$

$V_2, V_3, \ldots, V_d$ are however linearly independent.

Now recall that $V = \{(a_1, \ldots, a_d) \in \mathbb{C}^d \mid \sum a_i = 0\}$ is the natural module.

We just compute action of $\Sigma_d$. Suppose $g \in \Sigma_d$ with $g(j) = i$.

$$\text{Then} \quad gV_j = \sum_{\substack{\sigma \in \Sigma_d \\ \sigma(d)=j}} g\sigma - \sum_{\substack{\tau \in \Sigma_d \\ \tau(1)=j}} \tau = \sum_{\substack{x \in \Sigma_d \\ x(d)=i}} x - \sum_{\substack{y \in \Sigma_d \\ y(1)=i}} y = V_i.$$

Thus the map $V_j \to e_j - e_1 = (-1, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{C}^d$ is an $\cong$ from $S^{(d-1,1)}$ to $V$.

1. $t_1 = \dfrac{\overline{123}}{\underline{45}}$ $t_2 = \dfrac{\overline{124}}{\underline{35}}$ $t_3 = \dfrac{\overline{125}}{\underline{34}}$ $t_4 = \dfrac{\overline{134}}{\underline{25}}$ $t_5 = \dfrac{\overline{135}}{\underline{24}}$ $t_6 = \dfrac{\overline{145}}{\underline{23}}$

$t_7 = \dfrac{\overline{234}}{\underline{15}}$ $t_8 = \dfrac{\overline{235}}{\underline{14}}$ $t_9 = \dfrac{\overline{245}}{\underline{13}}$ $t_{10} = \dfrac{\overline{345}}{\underline{12}}$

2. We will denote tabloids in $M^{32}$ by last row,

$e_{t_1} = 45-15-24+12$

$e_{t_2} = 35-15-23+12$

$e_{t_3} = 34-14-23+12$

$e_{t_4} = 25-15-23+13$

$e_{t_5} = 24-14-23+13$

$e_{t_6} = 23-13-24+14$

$e_{t_7} = 15-25-13+23$

$e_{t_8} = 14-24-13+23$

$e_{t_9} = 13-23-14+24$

$e_{t_{10}} = 12-14-23+34$

3. A basis is given by $\{e_{t_1}, e_{t_2}, e_{t_3}, e_{t_4}, e_{t_5}\}$
They are clearly linearly independent (notice they have distinct 1st terms) Also

$e_{t_6} = -e_{t_5}$

$e_{t_7} = -e_{t_4}$

$e_{t_8} = e_{t_5}$

$e_{t_9} = -e_{t_4}$

$e_{t_{10}} = e_{t_3}$

4.

$(12)e_{\sigma_1} = 45-25-14+12 = e_{\sigma_1} - e_{\sigma_4}$

$(12)e_{\sigma_2} = 35-25-13+12 = e_{\sigma_2} - e_{\sigma_4}$

$(12)e_{\sigma_3} = 34-24-13+12 = e_{\sigma_3} - e_{\sigma_5}$

$(12)e_{\sigma_4} = 15-25-13+23 = -e_{\sigma_4}$

$(12)e_{\sigma_5} = 14-24-13+23 = -e_{\sigma_5}$

$(12345)\, e_{\sigma_1} = 15-12-35+23 = -e_{\sigma_2}$

$(12345)\, e_{\sigma_2} = 14-12-34+23 = -e_{\sigma_3}$

$(12345)\, e_{\sigma_3} = 45-25-34+23 = e_{\sigma_1} - e_{\sigma_3} - e_{\sigma_4}$

$(12345)\, e_{\sigma_4} = 13-12-34+24 = -e_{\sigma_3} + e_{\sigma_5}$

$(12345)\, e_{\sigma_5} = \cancel{25-35-24+24 = -e_{\sigma_2}+e_{\sigma_3} \quad -e_{\sigma_2}+e_{\sigma_3}+e_{\sigma_4}-e_{\sigma_5}}$

$35-25-34+24 = e_{\sigma_2} - e_{\sigma_3} + e_{\sigma_5}$

$$(12) \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 \end{pmatrix} \qquad (12345) \to \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

5. There are $n$ different $(n-1,1)$ tabloids

The $e_t$'s are

$1-2,$
$2-1,$
$3-1,$
$4-1,$
$\vdots$
$n-1$

where $a$ denotes $\begin{matrix} 1 2 \cdots a-1 \, a+1 \cdots \\ a \end{matrix}$

Thus $S^{n-1,1} = \langle 2-1, 3-1, 4-1, \ldots, n-1 \rangle$

Thus the map sending $a-1$ to $e_a - e_1 \in \mathbb{C}^n$

is an $\cong$ of $S^{n-1,1}$ onto

$$\{ (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum z_i = 0 \}$$

Thus $\mathbb{C}^n \cong S^{n-1,1} \oplus \mathbb{C}$