We can use characters to decompose these, the character of a perm rep is always the # of fixed points

	1	(12)	(123)	(1234)	$(12)(34)$
χ_{vertices}	8	0	2	0	0
χ_{edges}	12	2	0	0	0

Use (,) to get

$\chi_{\text{vertices}} = U \oplus V \oplus U' \oplus V'$

$\chi_{\text{edges}} = U \oplus V \oplus V \oplus V' \oplus U$

2.27 We just write the char table of S_4 but only the A_4 classes

	e (1)	(123) (8)	$(12)(34)$ (3)
$\mathbb{C}$	1	1	1
sgn	1	1	1
V	3	0	-1
$V\otimes$sgn	3	0	-1
W	2	-1	2

so $\mathbb{C} \cong$ sgn and $V \cong V\otimes$sgn as A_4 modules (obviously!)

$\mathbb{C}$, sgn are irreducible

$(V,V) = \frac{1}{12} \cdot 9 + 3 = 1$ so V, $V\otimes$sgn stay irreducible.

$(W,W) = \frac{1}{12}(4 + 8 + 12) = 2$

W is not irreducible as A_4 module.

2.29

Suppose α is not a class function.

Let $\alpha(g) \neq \alpha(hgh^{-1})$

Let V be the regular representation of G.

$$\varphi_{\alpha,V}(h^{-1}) = \sum_{g \in G} \alpha(g) g h^{-1}$$

$$(*) \quad h\varphi_{\alpha,V}(h^{-1}) = \sum_{g \in G} \alpha(g) hgh^{-1} = \cancel{\Sigma}$$

$$(**) \quad \varphi_{\alpha,V}(hh^{-1}) = \varphi_{\alpha,V}(e) = \sum_{g \in G} \alpha(g) g$$

so coef of hgh^{-1} in $(*)$ is $\alpha(g)$, in $(**)$ it is $\alpha(hgh^{-1})$.

Thus $h\varphi_{\alpha,V}(h^{-1}) \neq \varphi_{\alpha,V}(hh^{-1})$

so $\varphi_{\alpha,V}$ is not a G-homomorphism

2.33 a. Let $V = \bigoplus_{i=1}^{k} a_i V_i$ $\quad W = \bigoplus_{i=1}^{k} b_i V_i$ with V_i irreducibles

Then $(V,W) = (\sum a_i \chi_i, \sum b_j \chi_j) = \sum a_i b_i$

$$\mathrm{Hom}_G(V,W) = \bigoplus_{i,j=1}^{k} a_i b_j \mathrm{Hom}_G(V_i, V_j)$$

$$= \bigoplus_{i,j=1}^{k} a_i b_j \delta_i^j \text{ by Schur's lemma}$$

$$= \cancel{\oplus} \bigoplus_{i=1}^{k} a_i b_i \mathbb{C}$$

so $\dim \mathrm{Hom}_G(V,W) = (V,W)$

(b). Let $\chi = \sum a_i \chi_i$ with $a_i \in \mathbb{Z}$

Then $(\chi,\chi) = \sum a_i^2$ so $(\chi,\chi) = 1 \Rightarrow \chi = -\chi_i$ or $+\chi_i$ for some i.

$(\chi,\chi) = 2$ and $\chi(1) > 0 \Rightarrow \chi = \chi_i + \chi_j$ or $\chi_i - \chi_j$ $i \neq j$ is clear.

c. $U \in V \otimes W$ if and only if $(U, V \otimes W) \neq 0$

if and only if $\mathrm{Hom}_G(U, V \otimes W) \neq 0$

if and only if $\mathrm{Hom}_G(V^* \otimes U, W) \neq 0$

iff $\mathrm{Hom}_G(W, V^* \otimes U) \neq 0$

since both have dimension $= (W, V^* \otimes U)$

iff $W \in V^* \otimes U$.

$\dim(U \otimes V^*) = \dim U \dim V$ so if W is a summand, obviously $\dim W \leq \dim U \dim V$.

2.36

Suppose V_1, V_2 are irreducible. Then let $\chi = \chi_{V_1} \boxtimes \chi_{V_2}$

$$(\chi,\chi) = \frac{1}{|G_1 \times G_2|} \sum_{(g_1,g_2) \in G_1 \times G_2} \overline{\chi(g_1,g_2)}\, \chi(g_1,g_2)$$

$$= \frac{1}{|G_1|}\frac{1}{|G_2|} \sum_{(g_1,g_2)} \overline{\chi_1(g_1)}\,\overline{\chi_2(g_2)}\, \chi_1(g_1)\,\chi_2(g_2)$$

$$= \frac{1}{|G_1|} \sum_{g_1 \in G_1} \overline{\chi_1(g_1)}\,\chi_1(g_1) \cdot \frac{1}{|G_2|}\sum_{g_2 \in G_2} \overline{\chi_2(g_2)}\,\chi_2(g_2)$$

$$= (\chi_1,\chi_1) \cdot (\chi_2,\chi_2) = 1 \cdot 1 = 1$$

so χ is irreducible.

$$= \frac{1}{|G|} \sum_{\substack{\text{classes}\\ C}} |C|\,\psi(C) \cdot \sum_{n=0} (\Lambda^n(C)$$

2.36 (cont)

Now let's check we have all irred. chars of $G_1 \times G_2$ by checking the sum of the squares is $|G_1 \times G_2|$!

$$\sum_{\substack{\text{chars}\\ \chi_1 \otimes \chi_2}} \text{degree}^2 = \sum_{\substack{\chi \in Irr(G_1)\\ \chi_2 \in Irr(G_2)}} (\chi_1 \otimes \chi_2)(e,e)^2$$

$$= \sum_{\chi_1, \chi_2} \chi_1(e)^2 \chi_2(e)^2$$

$$= \Big(\sum_{\chi_1 \in Irr G_1} \chi_1(e)^2 \Big)\Big(\sum_{\chi_2 \in Irr G_2} \chi_2(e)^2 \Big)$$

$$= |G_1| \, |G_2| = |G_1 \times G_2|$$

2.37

Let V be faithful with character χ. Let φ be an arbitrary irreducible. The character of $V^{\otimes n}$ is χ^n so our goal is to prove that $a_n = (\varphi, \chi^n)$ is $\neq 0$ for some $n \geq 1$

Consider $\displaystyle\sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} (\varphi, \chi^n) t^n$

$$= \frac{1}{|G|} \sum_{n=0}^{\infty} \cdot \sum_{\substack{\text{classes}\\ c}} |c| \, \overline{\varphi(c)} \, \chi(c)^n t^n$$

we are summing over classes and weighting by $|c|$!

$$= \frac{1}{|G|} \sum_{\substack{\text{classes}\\ c}} |c| \, \overline{\varphi(c)} \cdot \sum_{n=0}^{\infty} (\chi(c) t)^n$$

Now $\frac{1}{1-x} = 1 + x + x^2 + \ldots$ so

$$\sum_{n=0}^{\infty} a_n t^n = \frac{1}{|G|} \sum_{C} \frac{|C| \overline{\varphi(C)}}{(1 - \chi(C) t)}$$

Now $\chi(C) = \dim V$ for $C = \{e\}$

but $\chi(C) \neq \dim V$ for any other class. (V is faithful!

Thus the RHS is not constant, so some $a_n \neq 0$ for $n > 0$.