Homework # 20- Due Tuesday 4/4/06, Assigned 3/23/06

1. (Proof of the "Five-Lemma"). Let:

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d, "\alpha_1"'] & U_2 \arrow[r] \arrow[d, "\alpha_2"'] & U_3 \arrow[r] \arrow[d, "\alpha_3"'] & U_4 \arrow[r] \arrow[d, "\alpha_4"'] & U_5 \arrow[d, "\alpha_5"'] \\ V_1 \arrow[r] & V_2 \arrow[r] & V_3 \arrow[r] & V_4 \arrow[r] & V_5 \end{tikzcd}$$

be a commutative diagram of A-module homomorphisms with exact rows. Prove:

a. α_1 is onto and α_2, α_4 1-1 $\rightarrow \alpha_3$ is 1-1.

b. α_5 is 1-1 and α_2, α_4 onto $\rightarrow \alpha_3$ is onto.

c. α_1 onto, α_5 1-1 and α_2, α_4 isomorphisms implies α_3 is an isomorphism.

2. Suppose $0 \rightarrow A \rightarrow B \xrightarrow{f} C \rightarrow 0$ and $0 \rightarrow C \xrightarrow{g} D \rightarrow E \rightarrow 0$ are short exact sequences or modules. Prove:

$$0 \rightarrow A \rightarrow B \xrightarrow{gf} D \rightarrow E \rightarrow 0$$

is exact. Show that every exact sequence may be obtained by splicing together suitable short exact sequences in this way.

3. Suppose $f : U \rightarrow V$ and $g : V \rightarrow U$ are A-module homomorphisms such that $gf = \mathrm{Id}_U$. Show that $V = \mathrm{Im}\, f \oplus \ker g$.