Homework # 18- Due Tuesday 4/4/06, Assigned 3/21/06

1. Let N be a normal subgroup of G. Prove:

$$\mathrm{rad}(kN) = kN \cap \mathrm{rad}(kG).$$

2. Let N be normal in G and let T be a simple kN-module. Show there exists a simple kG-module S withe T a direct summand of $\mathrm{Res}_N^G(S)$. (This is part of the Clifford correspondence).

3. For an algebra A define $[A, A]$ to be the subspace spanned by all elements of the form $\{ab - ba \mid a, b \in A\}$. Show that $[M_n(k), M_n(k)]$ is the set of all matrices of trace zero.

4. Let $G \cong C_p \times C_p =< x > \times < y >$ be as in class. Let W be a two-dimensional vector space over k with basis u, v.

a. For $\alpha, \beta \in k$, show that W has a kG-module structure such that:

$$\begin{aligned} xu &= u + \alpha v \\ xv &= v \\ yu &= u + \beta v \\ yv &= v \end{aligned}$$

Call this module $V_{(\alpha,\beta)}$.

b. Prove $V_{((\alpha,\beta)}$ is indecomposable unless $(\alpha, \beta) = (0, 0)$.

c. Prove that $V_{(\alpha,\beta)} \cong V_{(\gamma,\delta)}$ if and only if (α, β) and (γ, δ) are proportional.

d. Prove that any two-dimensional G module is isomorphic to some $V_{(\alpha,\beta)}$. Thus we have an entire projective line worth of isomorphism classes of two-dimensional indecomposable G-modules.

5. Prove that there are exactly two isomorphism classes of indecomposable 3-dimensional kG-modules V with $(x - 1)^2V = (y - 1)^2V = 0$.