

Math 6980 Homework # 11, Assigned 2/14/06, Due 2/21/06

1. Let a group G act on a set S . We say that G is *transitive* if, given any $s, t \in S$, there is a $g \in G$ with $gs = t$. The group is *doubly transitive* if, given any $(s, u), (t, v) \in S \times S$ with $s \neq u, t \neq v$, there is a $g \in G$ with $gs = t$ and $gu = v$.

a. Prove that the multiplicity of the trivial representation in the permutation module $\mathbf{C}S$ is the number of orbits in S under the action of G . Thus if G is transitive this multiplicity is one. (For example the permutation module M^μ).

b. Now assume the action is doubly transitive. Show that the complement to the trivial submodule of $\mathbf{C}S$ is irreducible, i.e. if $\mathbf{C}S$ has character χ then $\chi - 1$ is an irreducible character of G . Hint: Fix $s \in S$ and use Frobenius reciprocity on the stabilizer G_s of s .

c. Use part c to conclude that the function $f : \Sigma_d \rightarrow \mathbb{C}$ given by $f(\pi) = (\text{number of fixed points of } \pi) - 1$ is an irreducible character.

2. Let $s_i = (i, i + 1) \in \Sigma_d$ for $i = 1, 2, \dots, d - 1$. Verify the s_i satisfy the relations:

$$\begin{aligned} s_i^2 &= e \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \quad 1 \leq i \leq d - 2 \\ s_i s_j &= s_j s_i \quad |i - j| > 1 \end{aligned}$$

It can be shown the s_i generate Σ_d subject to these relations, which are known as Coxeter relations.

Fix a partition $\lambda \vdash d$ and fix an ordering of the standard λ -tableaux $t_1, t_2, \dots, t_{f_\lambda}$. We are going to write down the matrices for S^λ explicitly.

Let t be a λ -tableau and define the *axial distance* from k to $k + 1$ in t to be:

$$\delta_i = \delta_i(k, k + 1) = (c' - r') - (c - r)$$

where c, c' and r, r' are the column and row coordinates of k and $k + 1$ respectively, in t_i . *Young's seminormal form* assigns to each $s_k = (k, k + 1)$ the matrix $\rho_\lambda(s_k)$ with entries:

$$\rho_\lambda(s_k)_{i,j} = \begin{cases} 1/\delta_i & i = j \\ 1 - 1/\delta_i^2 & \text{if } s_k t_i = t_j, i < j \\ 1 & \text{if } s_k t_i = t_j, i > j \\ 0 & \text{else} \end{cases}$$

a. Show that every row and column of $\rho_\lambda(s_k)$ has at most two nonzero entries.

b. Show that ρ_λ can be extended to a representation of Σ_d by showing the matrices $\rho_\lambda(s_k)$ satisfy the relations above.

Remark: This representation is equivalent to S^λ .